Early Family Math

Chapter 3 Bonus Material

— Introduction —

Are you someone who wishes there were more examples, discussions, and commentaries in the intentionally brief descriptions of the lessons? If so, you have come to the right place! This file contains bonus material for some of the activities from chapter 3.

For puzzles, many examples of solved puzzles are given, along with additional commentary on how to create them. The Early Family Math program is based on the idea that early mathematics is something a family should do together, and making puzzles for your child to do with you is an important part of that process. Once you get the hang of each puzzle, you should find that most if not all the puzzles are fairly easy for you to create.

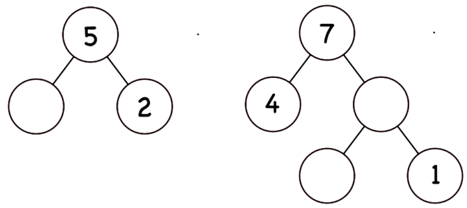
Many of these puzzles have different levels of difficulty, and there are many suggestions and examples in the coming pages for how to create those levels. Always start with the easiest puzzles. It is far better to have your child experience success, understanding, and fun with puzzles that are a bit too easy, than to be frustrated, discouraged, and over-challenged by puzzles that are too hard. Once your child builds confidence and enthusiasm for a math activity, that is the time to slowly incorporate greater challenges. Also, not all puzzles will be fun for everyone, so don't push puzzles and activities that just don't seem to connect.

This is what you will find in the following pages:

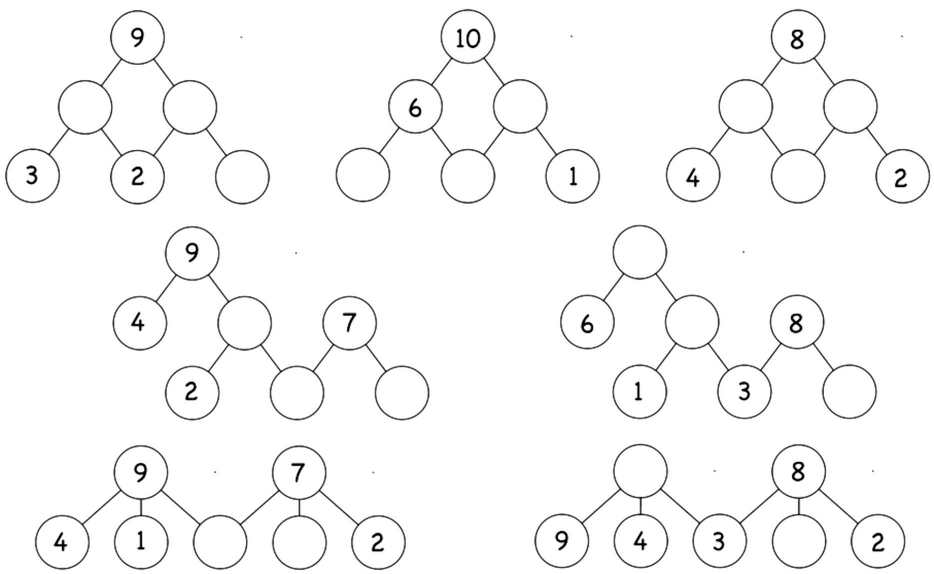
Chapter 3 - Shape Sums

These puzzles use numbered circles connected in an upward fashion, and every circle is the sum of all the circles directly below and connected to it.

The easiest puzzles have most of the circles filled in. Here are two examples that are straightforward to solve.



These puzzles can be made more difficult by having one circle used in more than one direction. All of the next seven puzzles are direct calculations except the rightmost one of the first row. It is trickier because the one circle in the middle is shared by two unknown circles above it. That puzzle involves small enough numbers that it can be solved easily with a little trial and error.

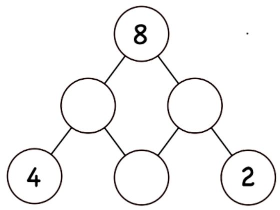


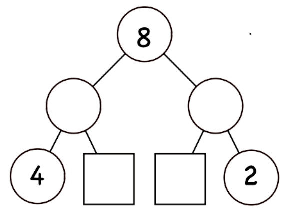
Another option for adding complexity to these puzzles is to use non-circular shapes. While the value in a circle may or may not duplicate the value in some other circle or shape, the value in a non-circular shape must match the value in all other places with the same shape. For example, all squares have the same value. Use matching shapes to practice adding twins, near twins, and halving.

If you like, you can add the rule that two non-circular shapes that have different shapes must have different values - for example, a square and a hexagon would have to have different values.

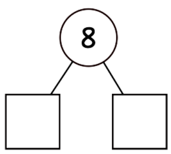
Make any of these puzzles by starting with a diagram that is completely filled in and then removing some numbers. If the puzzle has some repeated numbers, use a square or other shape instead of a circle for that repeated number.

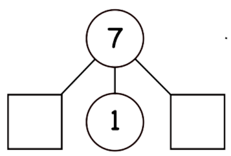
The next two puzzles illustrate the psychological difference between using a circle from two directions and replacing the circle with two squares. These two puzzles are essentially the same, but a young child will find the first one much easier to understand and work with. Please give your child plenty of practice with circle-only puzzles before venturing into more sophisticated puzzles with non-circular shapes.

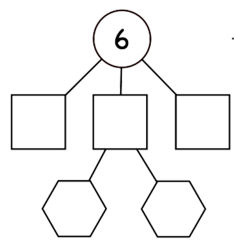




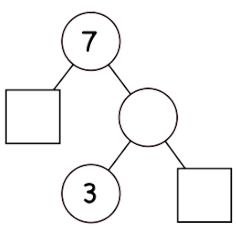
Puzzles similar to the next three are useful for practicing adding twins, near twins, and triples.

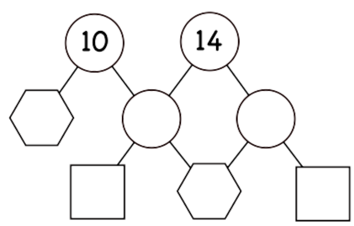


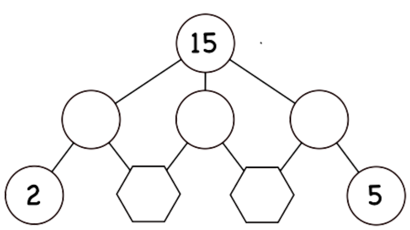


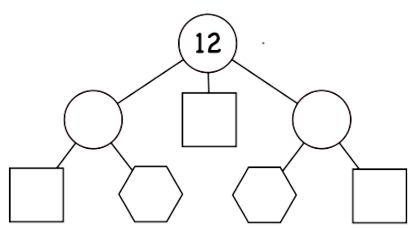


Here are some examples of using non-circular shapes to make trickier puzzles. If your child enjoys these, there are a great many more variations to explore. Happy puzzling!









Chapter 3 - Nim Doubling the Limit

— One Pile —

Set a starting total, say 20. Let your child choose whether to go first or second. During the first turn, a player chooses to subtract 1 or 2 from the current total. After the first turn, a player may subtract any number from 1 up to twice the number used on the last turn. The first person to reach 0 wins.

There are many alternative versions of this game. Some of them are:

- The first person to reach the target loses.
- Instead of using the range of 1 to 2, the initial range is from 1 to one less (or two less) than the target number.
- Practice adding, rather than subtracting, by starting at 0 and having the first person to reach the target win (or lose).
- The initial limit is one (or two) less than the target number, and instead of doubling the value used of the last turn, use the value of the last turn as the limit.
- The initial limit is one (or two) less than the target number, and instead of doubling the value used of the last turn, use the triple of the value of the last turn.

As you can see, there are lots of variations. Make up your own family rules if you are enjoying the game.

For the most part, these games are much harder to analyze than the versions of Nim that use a fixed set of choices for each move.

— More Than One Pile —

Yet another way to make new versions of this game is to use more than one number. Picture this version as having several piles of tokens (pebbles, bits of food). For example, you could have two piles with 12 tokens in one pile and 8 in the other. A standard rule to use is that you can take any number of tokens, but they must all be from one pile.

Alternative versions of this game are:

- There are more than two piles.
- You have the option of taking the same number of tokens from all the piles.
- You have the option of taking the same number of tokens from the piles you choose.
- You can only take tokens from the largest pile.

As you can imagine, there are even more versions of this game; however, perhaps this is more than enough for now!

Chapter 3 - Counting Evens and Odds

— Basic Setup —

Use a small collection of Number Cards involving some small quantities. Start with three cards and later on use more cards if your child enjoys the investigation..

Suppose the numbers are 1, 2, and 3. The question is: If you randomly pick two cards and add them, are you more likely to get an even number or an odd number?

There are two ways to look into this. One way is to do experiments. Shuffle the cards, randomly pick two cards, and see whether the sum is even or odd. After each experiment, put a tick mark in the appropriate column on a piece of paper to count the even and odd results.

The second way is to count how many ways there are of getting an odd number versus an even number. For example, in the case of using 1, 2, and 3, there is one way to get an even number (1 + 3) and two ways to get an odd number (1 + 2, 2 + 3). So, for the numbers 1, 2, and 3, the odd number sums are twice as likely.

After you've played around with 1, 2, and 3 for a while, try other groups of three cards. Does 2, 3, and 4 behave any differently? The groups 1, 3, 5 and 2, 4, 6 produce only even numbers - why is that? After playing around with three cards for a while, see what happens with 4 or more cards.

To make a game of it, let one player be Even and the other player be Odd. See who has the most successes after a dozen trial runs.

— Investigation Analysis —

The fun thing about an investigation is that it invites a person to play with the numbers and be a mathematician. As mentioned above, play around with different groups of three numbers. After some experimentation, your child may notice that any group of three numbers that has at least one even number and one odd number behaves the same. However, if all the numbers are all odd numbers or all even numbers, then the sums are all even. Which brings up the usual question: Why does that happen?

After some experimentation, even a young child can stumble upon the beautiful number theory rule that says:

- Even plus Even is Even
- Even plus Odd is Odd
- Odd plus Odd is Even

Why does this rule work? Use the Number Shapes activity to represent even numbers and odd numbers with two rows of tokens - when will adding these numbers come out to two equal rows?

Once this rule is discovered, your child may realize that the particular numbers do not matter so much. Having the numbers 1, 2, 3 is really no different from having the numbers 3, 4, 5 (or 3, 12, 17 for that matter). The analysis really depends on how many numbers are even and how many are odd.

With that in mind, here is a table of the possible outcomes for groups of size three and four.

3 Numbers:

- 3 Evens, 0 Odds - 3 Even sums
- 2 Evens, 1 Odd - 1 Even sum, 2 Odd sums
- 1 Even, 2 Odds - 1 Even sum, 2 Odd sums
- 0 Evens, 3 Odds - 3 Even sums

4 Numbers:

- 4 Evens, 0 Odds - 6 Even sums
- 3 Evens, 1 Odd - 3 Even sums, 3 Odd sums
- 2 Evens, 2 Odds - 2 Even sums, 4 Odd sums
- 1 Even, 3 Odds - 3 Even sums, 3 Odd sums
- 0 Evens, 4 Odds - 6 Even sums

The results are surprising and leave many things to investigate if one is interested! What happens with 5 numbers, 6 numbers, or more? Why is it that interchanging Even numbers and Odd numbers does not seem to change the results? For example, if you have 3 Evens and 1 Odd you get the same results as 1 Even and 3 odds. For circumstances like 3 Evens and 1 Odd, why do the results come out balanced when the Even and Odd counts start out unbalanced?

This is some cool mathematics and even a small child can play around with it!

Chapter 3 - Sum Groups

These puzzles use a grid of numbers with a target sum. Find groups of two, three, or four numbers that add up to the target. The members of a group must share sides. Use tokens, such as different types of food items, to identify each group within the puzzle. When complete, the entire puzzle will be made up of identified groups.

6

1	2	2
5	3	4
1	3	3

8

0	8	3	2
2	4	4	3
6	5	5	7
1	2	3	1

These puzzles provide particularly good practice with number bonds. By using tokens instead of a pencil, you can use puzzle sheets over and over.

Create these puzzles by starting with an empty grid and putting in numbers around the grid using pairs and triples that add up to the target sum. It's more fun if the puzzle has just one solution, but don't worry about it.

6

1	2	2
5	3	4
1	3	3

1	6	2
1	0	4
4	1	5

1	2	3
5	3	4
1	3	2

4	2	1
3	5	1
3	1	4

1	0	1
5	5	4
3	3	2

6

5	1	4	2
3	1	3	3
2	2	3	1
5	1	4	2

4	5	1	3
2	1	3	3
5	2	2	4
1	3	1	2

1	5	2	4
3	2	3	2
1	1	2	4
3	3	5	1

1	5	2	1
3	2	1	5
1	2	3	1
2	4	3	3

7

2	4	3
5	2	1
6	1	4

2	6	1
1	4	5
4	3	2

7	1	3
0	3	4
1	6	3

5	1	1
4	4	3
3	7	0

4	4	3
1	2	2
6	1	5

7

5	2	1	1
6	1	2	6
3	4	3	1
4	3	5	2

6	1	4	1
4	5	2	3
3	2	3	4
1	6	3	1

4	5	2	1
3	1	3	4
2	3	4	2
3	2	2	1

2	5	3	4
1	5	4	3
6	2	1	6
6	1	2	5

8

5	1	7
1	2	3
6	2	5

6	2	4
3	1	4
5	3	4

4	4	1
4	2	7
2	3	5

7	1	0
1	2	8
5	3	5

1	0	4
4	8	4
3	6	2

8

0	8	3	2
2	4	4	3
6	5	5	7
1	2	3	1

2	3	5	3
6	4	3	2
2	4	3	5
4	2	1	7

2	3	2	1
3	2	5	2
1	6	1	3
7	4	4	2

7	1	2	3
2	1	6	5
3	5	1	3
5	4	4	4

9

1	0	9
4	6	5
4	3	4

5	6	3
4	5	7
3	1	2

1	2	7
3	5	4
0	9	5

4	1	8
2	3	3
5	4	6

7	4	5
2	6	2
1	8	1

9

5	4	3	6
7	4	2	3
2	5	3	6
8	1	1	3

5	5	4	5
2	4	2	7
2	6	3	6
1	8	1	2

5	2	2	1
3	5	2	6
3	1	3	4
3	7	2	5

2	3	6	3
7	5	3	3
2	2	7	2
5	4	1	8

10

8	2	3
5	3	4
5	7	3

6	5	5
1	3	6
2	8	4

7	5	4
3	1	9
4	6	1

4	2	1
4	5	3
4	1	6

1	9	7
4	3	3
3	4	6

10

1	5	3	2
4	3	7	4
5	3	5	6
3	4	1	4

8	9	1	3
1	1	3	4
6	3	5	5
4	7	1	9

4	1	5	5
5	3	2	1
6	5	7	2
4	1	6	3

1	6	8	2
3	1	3	6
3	1	6	5
7	9	4	5

Chapter 3 - Zoo Rescue

— Game Description —

In this game, use two dice or two sets of number cards going from 1 to 6. Each player has 6 tokens – animal tokens are perfect for this game if you have them. Each player also has a piece of paper with boxes numbered from 0 to 5. Each player decides where to put their 6 tokens – it is okay to put more than one token in a box.

During a player's turn, two numbers are created by rolling the dice or picking two cards, and the difference of those numbers is used. A player can free one of their tokens if they have one in that box. The first player to rescue all their tokens wins.

— Strategy for Placing Tokens —

How should a player place the 6 tokens? As is often a good idea, let's start with a simpler question: Where would the best place be to put 1 token. This would clearly be in the box most likely to occur. Rather than doing any tricky analysis, we can simply list out the possibilities and see which differences happen the most.

1-1	0		2-1	1		3-1	2		4-1	3		5-1	4		6-1	5
1-2	1		2-2	0		3-2	1		4-2	2		5-2	3		6-2	4
1-3	2		2-3	1		3-3	0		4-3	1		5-3	2		6-3	3
1-4	3		2-4	2		3-4	1		4-4	0		5-4	1		6-4	2
1-5	4		2-5	3		3-5	2		4-5	1		5-5	0		6-5	1
1-6	5		2-6	4		3-6	3		4-6	2		5-6	1		6-6	0

Counting up the results, we have 0 - 6, 1 - 10, 2 - 8, 3 - 6, 4 - 4, 5 - 2. So, 1 is clearly the best choice and it will happen 10 / 36 of the time. We can rank them in order of frequency as 1, 2, 3, 0, 4, and 5.

The much harder question is what to do with more than one token. Once you've seen these numbers, a good question for an older child is: why wouldn't you just put all your tokens on 1? To see the answer to this, imagine the simpler situation where you had only two tokens and you ignored all results that weren't 1 or 2. Then 1 would happen 10 / 18 of the time and 2 would happen 8 / 18 of the time. If you put both tokens on 1, you would need to get a 1 and then a 1 to win after two rolls. However, if you put a token on 1 and a token on 2, you would be successful after two rolls with a 1 and then a 2, or a 2 and then a 1 - something that is about 60% more likely to happen!

Rather than go into a long, detailed analysis, let's just leave it at something fairly simple that appeals to our intuition - put most of your tokens on 1, the second most on 2, and maybe one on 0 or 3. There's no guarantee you'll win, but you should do pretty well in the long run!

Chapter 3 - Common Sums

— Investigation Introduction —

Make a sheet of paper with 12 rows. In each row, put 8 squares. The leftmost column of squares has the numbers from 1 to 12 written in the squares. Put 1 token on each of the 12 numbers. Start rolling a pair of dice. After each roll, move the token for the sum of the dice one square to the right. The goal for each token is to be the first to get all the way to the right across the page.

Let your child come up with some questions to investigate. Some natural questions are:

- Which token will win and why?
- Which tokens do well and which ones do poorly?
- Which token is the worst?
- How will the winners change if the rows are changed to have fewer squares or more squares?

Have your child explain their ideas about the answers to these questions, and then investigate their ideas by running experiments.

Add a competitive element to this by guessing which token will win before the round starts.

— Analysis —

As with the analysis of the previous game, the simplest way to analyze this is to list out all the possibilities.

1+1	2		2+1	3		3+1	4		4+1	5		5+1	6		6+1	7
1+2	3		2+2	4		3+2	5		4+2	6		5+2	7		6+2	8
1+3	4		2+3	5		3+3	6		4+3	7		5+3	8		6+3	9
1+4	5		2+4	6		3+4	7		4+4	8		5+4	9		6+4	10
1+5	6		2+5	7		3+5	8		4+5	9		5+5	10		6+5	11
1+6	7		2+6	8		3+6	9		4+6	10		5+6	11		6+6	12

Summarizing the frequency we have: 1 - 0, 2 - 1, 3 - 2, 4 - 3, 5 - 4, 6 - 5, 7 - 6, 8 - 5, 9 - 4, 10 - 3, 11 - 2, 12 - 1. By the way, these are good numbers to remember for any dice game that involves summing the two dice!

So, 1 will always lose and 7 is the most likely to win. However, the difference in frequency between 7 and 6 or 8 is not very great. If you just do a few rolls, it would be very hard to predict with any certainty which one would win. It is only when you do a great many rolls that you can guarantee that 7 will win eventually.

Chapter 3 - Sudoku Variations

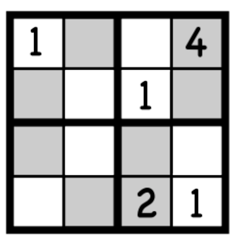
There are a great many Sudoku variations in the world, and there are even more other puzzles that are similar to those Sudoku variations. This section will look at five of these Sudoku variations. These all follow the rule of the “Latin Square” - that every number occurs exactly once in each row and column.

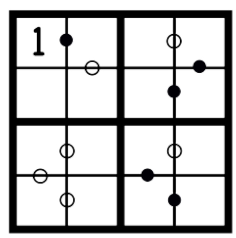
You can make any of these Sudokus by starting with a filled in puzzle of the appropriate type - either a Latin Square or a Jigsaw Sudoku. All of the Sudoku solutions given in the Bonus Material for Chapters 1-2 should be of use to you for this. After you have a solution in hand, add the additional information needed for this special kind of puzzle and remove some or all of the numbers.

— Jigsaw Sudokus With Extra Information —

These two puzzle types are Latin Squares that have the additional restriction that each subregion has every number occurring in it exactly once. In addition to being a Jigsaw Sudoku, they have additional properties.

Even-Odd Sudokus. In these puzzles, the even numbers are greyed in. This additional information tends to make these puzzles very easy and it is usually possible to remove almost all the numbers.



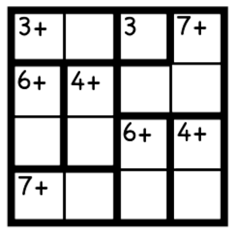


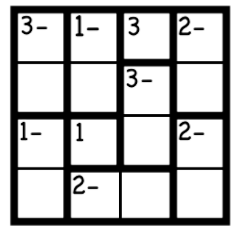
Kropki Sudokus. This is the same as regular Sudoku except two types of dots placed between cells are added. If the dot is hollow, then the two numbers are one apart. If the dot is filled in, then one number is half the other number. Similar to Even-Odd puzzles, this additional information tends to make these puzzles pretty easy and that means that almost all numbers can be removed.

— Sudokus With Adding and Subtracting —

These puzzles are broken into subregions that have a target number assigned to them. Unlike standard Sudoku, it is allowed for a number to be repeated in a subregion as long as the puzzle is still a Latin Square. If a subregion has just one square in it, then the target number will be the value of that square.

In a Sumdoku Sudoku puzzle, the sum of all the numbers in a subregion is the given target number. In a Diffdoku Sudoku puzzle, all subregions have one or two squares. If a subregion has two squares, then the difference of the two numbers is the given target number.





In a Sumdiffdoku Sudoku puzzle, both addition and subtraction are used. The subregions are marked with a “+” or a “-” to indicate whether to take a sum or difference.

The three types of puzzles are usually made with no numbers given in them. Of course, the subregions with one square are essentially squares with the number filled in. For a young child, you may want to supply quite a few of the numbers to make the puzzle within their sophistication level.

To vary the math calculations, use different groups of numbers instead of the usual 1 to 4 for a 4 by 4. For example, use the numbers 1, 3, 5, and 7. If you do this, list the numbers above the puzzle so your child will know what to use.

Chapter 3 - How Many Ways

Counting the number of ways of making choices can lead to some interesting results. Most of these counting situations benefit from being looked at systematically. This is hard for a child to do, and that's okay - let them play around with it and enjoy the exploration. Being systematic can wait until they are older.

— Investigation 1 —

Drawing with only red and blue, how many ways can you draw a monster with a hat, eyes, and cape? How does this change if you only colored the hat and the cape? How would it change if you used three colors, or if you could only use each color once?

To do this investigation in a sophisticated way involves multiplication, and it is too soon for that. However, your child can play around with these ideas and start developing a sense for how to do this kind of counting.

Let's tackle these questions one at a time. The hat can be either red or blue, the eyes can be either red or blue, and the cape can be either red or blue. Each object to color doubles the number of possibilities. Thus, 2 doubled and then doubled again gives 8 possibilities. Listing these out is a good way to see it. Let R be for red and B be for blue, and list the colors in the order for the hat, the eyes, and the cape. The possibilities are: RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB.

Coloring only the hat and cape gives 2 doubled, which is 4 possibilities. The list for this is: RR, RB, BR, BB.

If you had three colors for the three things to color, you would have 3 x 3 x 3 = 27 possibilities (a long list).

In general, if you have events that don't influence each other, multiply the possibilities. If you are only allowed to use each color once, the events restrict each other and do influence each other. Let's list them out using G (for green) for the third color: RBG, RGB, BGR, BRG, GRB, GBR.

— Investigation 2 —

You have a row of 5 identical candies. How many ways can you color them to give 2 red ones and 3 blue ones?

Mark 2 pieces of paper with an R and 3 pieces of paper with a B. Your child can play with the ten ways there are to lay these out. The list is: RRBBB, RBRBB, RBBRB, RBBBR, BRRBB, BRBRB, BRBBR, BBRRB, BBRBR, BBBRR. One way to look at this is that once you decide the 2 spots for red, blue has no choice and must go into the other 3 spots. Interestingly, you can also look at it the other way as placing the 3 blue pieces first.

If you're having fun, vary this investigation by changing the three numbers - just make sure the two smaller numbers add up to the total number of candies.

— Investigation 3 —

Find all the ways to get a sum by adding the numbers 1 and 2. Do this with and without considering order.

Don't consider order. Look at the example of adding up to 4. The possibilities are 1+1+1+1, 2+1+1, and 2+2. There are 3 ways to do this. After trying a few more examples, you realize that you are counting the number of ways of using 2's to add up to numbers less than or equal to 4. You can have 0 to 2 of the 2's, so there are 3 ways to do it. In general, the answer will be one more than half the number for even numbers, and one more than half of one less than the number for odd numbers.

Consider order. For the example of 4, the possibilities are 1+1+1+1, 2+1+1, 1+2+1, 1+1+2, and 2+2. So there are 5 ways to do it. Play around with lots of examples and make a table of the results. Here is what you should get (okay, you probably didn't go up to 10):

1	2	3	4	5	6	7	8	9	10
1	2	3	5	8	13	21	34	55	89

After looking at these numbers, your child may notice that each pair of numbers adds up to the next number. Why does this happen? These numbers are called Fibonacci Numbers and they show up surprisingly often.

To see why these numbers occur in this investigation look at the example of 4 and look at the last number used in the sum. The last number is either 1 or 2. If it is a 1, then the previous numbers give all the ways of adding up to 3. If the last number is a 2, then the previous numbers give all the ways of adding up to 2. So, the number of ways of adding up to 4 is the total of the ways of adding up to 3 plus the ways of adding up to 2.

Bigger numbers. If you are enjoying this, you can play around with the number of ways of getting sums that involve the numbers from 1 to 3 or even 1 to 4. Looking for patterns in these cases is much harder, but playing with the numbers will be just as fun.

Chapter 3 - Card Deck Ordering

— Introduction —

The challenge is to stack a deck of numbered cards, say 1 to 5, so that the following is true:

The top card is 1. Set aside this top card. Move the next card to the bottom of the deck. The next card is 2 and is set aside. Move the next card to the bottom of the deck. Continue until all cards are set aside in order.

Once your child finds it easy for 1 to 5, challenge your child to do it for larger number ranges.

— Be Systematic —

The difficulty with this puzzle is being systematic. For any size deck of cards, you can play around with it and eventually come up with the answer. Let's look for interesting patterns that make it easier.

Suppose you lay out the cards in order on the table. Here are the solutions for the first few cases. The numbers listed after the arrow give the order of the remaining cards after the first pass through the cards.

1

1 2 -> 2

1 3 2 -> 3

1 3 2 4 -> 3 4

1 5 2 4 3 -> 5 4

1 4 2 6 3 5 -> 4 6 5

1 6 2 5 3 7 4 -> 6 5 7

If there are an even number of cards (say 6), then the odd positions are filled with the first half of the cards in order (3 in this case), and the other spots are filled using the solution for half as many cards only bumped up in value. In the example for 6, the odd spots are filled with 1, 2, 3, and the even spots are filled with 4, 6, 5 - the values 1, 3, 2 (the solution for a three-card deck) each increased by 3.

The pattern for an odd number of cards is a little trickier. As before, the odd spots are filled with the first roughly half of the numbers (1 to 4 in the case of 7). If you look at the examples, the first card after the arrow is going to be moved to the end, so it should be the card you want last in that sequence. After that observation, the answer proceeds as in the even case.

Chapter 3 - Difference Pyramid

— Introduction —

The challenge is to place the numbers from 1 to 6 in a pyramid with one card in the top row, two cards in the second row and three cards in the third row, where each number is the difference of the two numbers below it.

If you are having trouble, here are two tips that help. The 6 must be in the bottom row because it cannot be the difference of any pair of numbers. Similarly, the 5 must either be in the bottom row or in the middle row above the 6 and the 1.

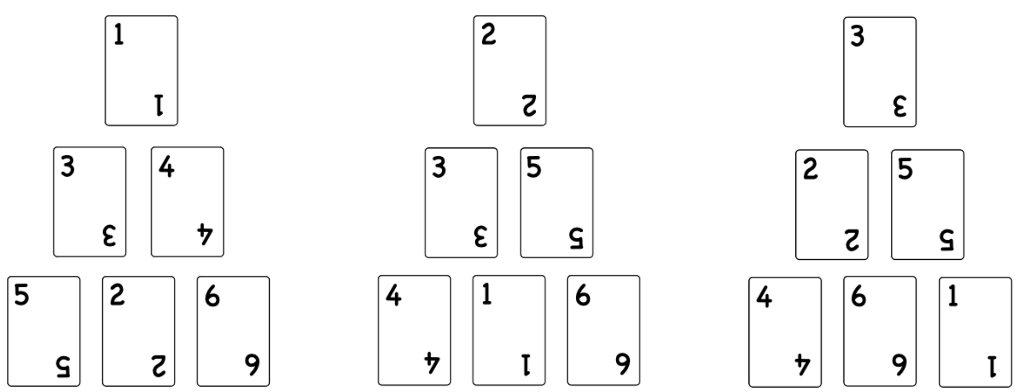
— What Are “Different” Solutions? —

If your child finds this puzzle easy to do, challenge them to find all the ways it can be done. Discuss what it means for two solutions to be different - if one solution is the mirror image of another, should it be considered different?

Answering the question of what makes solutions different is useful to do at the start. Because the mirror image of any solution is easy to make and is also a solution, it makes sense to ignore those. Ignoring mirror images will reduce the number of solutions to consider by half.

For example, we can assume that not only is the 6 in the bottom row, but it is either in the middle or the right side of the bottom row. Continuing that thinking with the 5, the bottom row can only have four possible layouts: 5 a 6, b 5 6, c 1 6, or d 6 1.

At this point it is a matter of working through the various possible values of a, b, c, and d. After some trial and error you will find that a is 2, b can never work, c must be 4, and d must be 4. So, ignoring mirror images, there are exactly three solutions:



— Larger Pyramids —

Let's use the cards from 1 to 10 to make a pyramid with four rows. This is a lot more complicated. A few cards can be placed, but after that it requires some determination. Because 10 cannot be the difference of two cards, it must go on the bottom row. Similarly, either 9 is in the bottom row or it is in the next-to-the-bottom row above the 1 and the 10. The 8 and 7 cards are also good cards to use to get rid of possibilities.

This means the bottom row looks like one of the following (ignoring mirror images):

a b 9 10, c 9 d 10, 9 e f 10, g h 10 9, i 9 10 j, 9 k 10 L, m n 1 10, o 1 10 p, q r 10 1

That is a lot of possibilities to consider!

Fortunately, if you consider where 8 and 7 can go, the possibilities are reduced to the following list (assuming there are no mistakes!). It is easy to finish each one of these after you have the bottom row.

8 3 10 9, 6 1 10 8, 8 1 10 6

Pyramids of size 15, 21, or higher are left to the truly dedicated. Good luck and enjoy!