Chapter 5 Bonus Material

— Introduction —

Are you someone who wishes there were more examples, discussions, and commentaries in the intentionally brief descriptions of the lessons? If so, you have come to the right place! This file contains bonus material for some of the activities from Chapter 5.

For puzzles, many examples of solved puzzles are given, along with additional commentary on how to create them. The Early Family Math program is based on the idea that early mathematics is something a family should do together, and making puzzles for your child to do with you is an important part of that process. Once you get the hang of each puzzle, you should find that most if not all the puzzles are fairly easy for you to create.

Many of these puzzles have different levels of difficulty, and there are many suggestions and examples in the coming pages for how to create those levels. Always start with the easiest puzzles. It is far better to have your child experience success, understanding, and fun with puzzles that are a bit too easy, than to be frustrated, discouraged, and over-challenged by puzzles that are too hard. Once your child builds confidence and enthusiasm for a math activity, that is the time to slowly incorporate greater challenges. Also, not all puzzles will be fun for everyone, so don't push puzzles and activities that just don't seem to connect.

This is what you will find in the following pages:

- **Chapter 5 – Nim with Factors**
- **Chapter 5 – Sieve of Eratosthenes**
- **Chapter 5 – Levers and Mobiles**
- **Chapter 5 – Divide Up the Box**
- **Chapter 5 – Letter Substitution Puzzles**
- **Chapter 5 – Investigations - Playing with Shapes**
- **Chapter 5 – Product Game**
- **Chapter 5 – Limited Calculators**
- **Chapter 5 – Double or Nothing**

Chapter 5 – Nim with Factors

— Introduction —

Start with any number, say 20. Let the child decide whether to go first or second. During their turn, a player may subtract any divisor of the current number from the number. The player forced to 0 loses.

— Analysis —

As usual, a good strategy for learning about this game is to look at a simpler version of the game, which in this case means starting with very small numbers. If it is your turn and you are faced with each of these numbers, here is what will happen: 1 - lose, 2 - win, 3 - lose, 4 - win, 5 - lose, 6 - win, 7 lose, and 8 win. By now the pattern is clear - if it is your move and you have an odd number, then you will lose; if you have an even number, then you will win.

Finding the winning strategy is a big step, but let's go deeper. Why does this work? What are the properties of odd and even numbers that create this situation? Set this question before your child and give them a lot of time to think about it and experiment with it - there is no hurry, and this process of wrestling with a question is invaluable and should not be short circuited.

Some experimentation with small numbers quickly reveals what is going on. If you have an odd number, all of the divisors are odd, so when you subtract any divisor the result is an even number. Consequently, odd numbers on one turn always lead to an even number on the next turn. Even numbers always have both odd and even numbers for divisors. So, the situation is not quite the same. However, if you have an even number, your goal is to give your opponent an odd number, and there is an easy way to do that - select the divisor 1 and subtract it!

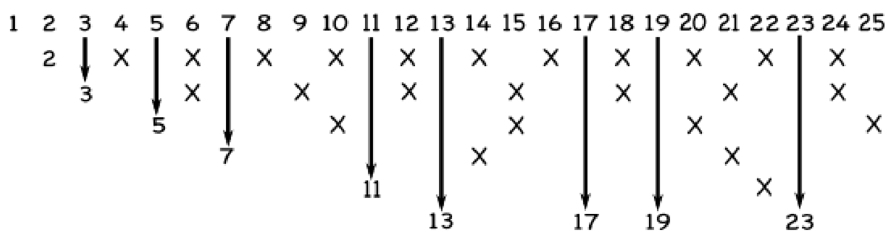
Chapter 5 – Sieve of Eratosthenes

— Introduction —

Start with a number line numbered from 1 to 25 - or a larger range if space and your patience allows.

Write the number 2 below itself. On the line even with this 2, put X's below each multiple of 2.

Now, pull down the first number with no X's below it (3 in this case) and put it on the next line. Write the 3 and put X's on that line for all its multiples. Continue in this way. At the end, you will have pulled down all the *primes*. Remember that 1 is a *unit* and not a prime!



— Analysis —

This simple process reveals some interesting facts about primes. See if your child can come up with some of these questions - however, if they don't arise naturally, here are some questions to ask.

1) Why are the numbers that drop down primes?

Suppose you have a composite number. We want to show that this number will have an X under it. Being composite, it is divisible by some number, n, between 1 and that number. If n is a prime, then our composite number would have an X under it from n being an earlier prime. If n is not a prime, then it has an X under it from some earlier prime, call it p. Now, p evenly divides n and n evenly divides our new number, so p must divide our new number. Consequently, when marking the multiples of p, an X would have been placed under our new number.

2) When you are placing X's for the multiples of a prime, there are some numbers that already have an X from an earlier prime. When does that happen and when doesn't it happen?

Let's look at the multiples of 5 in the sieve above. The multiples 5 x 2, 5 x 3, and 5 x 4 are already crossed out. Only 5 x 5 is new. This happens because 5 x 2, 5 x 3, and 5 x 4 are all multiples of 2 and 3, earlier primes. If we want to put X's in fresh places, we must multiply 5 by numbers that only have prime factors that are 5 and above. Because it is a little tedious to keep track of all that, what some people do is only cross out odd multiples and leave it at that.

3) For this sieve, what was the last prime that had a useful new X in its row?

In this sieve, the primes with useful X's are 2, 3, and 5. The multiples of 7 and 11 were all old X's. If you look at the answer to the last question, you will see the answer here. The only way to get new X's is to multiply a prime by primes bigger than or equal to itself. Once we reach a prime like 7 where $7 \times 7 > 25$, we do not need to check it. So, we only need to check primes whose square is smaller than or equal to the last number.

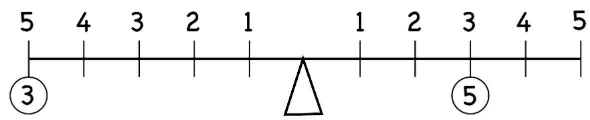
4) If you were given a number, say 53, which primes would you need to divide it by to see that it is prime?

From the answer to the last question, we only need to check primes whose square is less than or equal to 53. Those primes are 2, 3, 5, and 7 – none of these divides 53 evenly, so 53 must be prime!

Chapter 5 – Levers and Mobiles

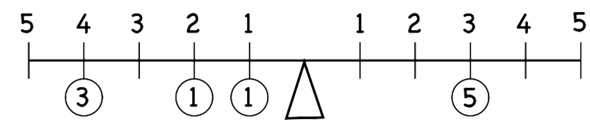
— Levers —

The lever principle states that the force exerted on one side of a lever by a mass is equal to the mass times its distance from the pivot point, the fulcrum.



In the lever above, the 3 on the left side is a distance of 5 from the fulcrum, so its force is 3 x 5 = 15. The 5 on the right side is a distance of 3 from the fulcrum, so its force is 5 x 3 = 15. This lever is in balance.

If there is more than one weight on a side, the forces will add up.



In this lever, there is 3 x 4 + 1 x 2 + 1 x 1 = 15 on the left side, and 5 x 3 = 15 on the right side. So it is in balance.

We will restrict these problems to only use whole numbers. You can decide whether you allow multiple weights to be hung off of the same point – we will assume it's okay to do multiple weights in the discussion that follows.

— Lever Puzzles —

You have a 3-unit weight and a 5-unit weight to put on opposite sides of the fulcrum. Where should they be put to balance? The answer to this can be the distances 5 and 3, but it can also be 10 and 6, or even larger answers such as 15 and 9. Be open to discussing whatever your child comes up with.

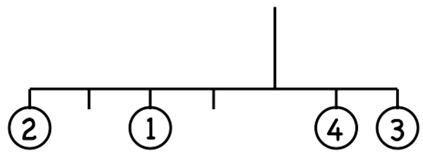
If you have a 3-unit and a 5-unit weight to put on one side of a lever, which weights can you put at which distances on the other side? This question continues the questions on the Make It Count page at the end of Chapter 4. As before, explore different combinations of weights. What happens if 3 and 5 are replaced by 4 and 5, 4 and 9, or 6 and 9?

How does this last problem change if we put the 3-unit and 5-unit weights on opposite sides of the fulcrum? Now it is easy to weigh a 1-unit weight by using 3 x 2 = 5 x 1 + 1 x 1. What other weights can you weigh this way?

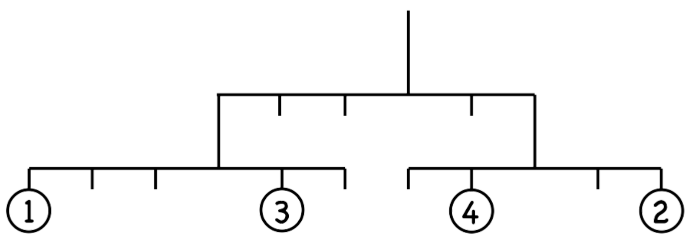
— Mobiles —

You are given some weights and a design for a mobile that has some attach points. The challenge is to put at most one weight per attach point so the mobile will balance along every arm. For the sake of these problems, we will assume the wires that create the mobile are weightless. Each arm in the mobile is a lever that needs balancing, so these puzzles are an extension of the Lever Balance - practice those puzzles before starting these.

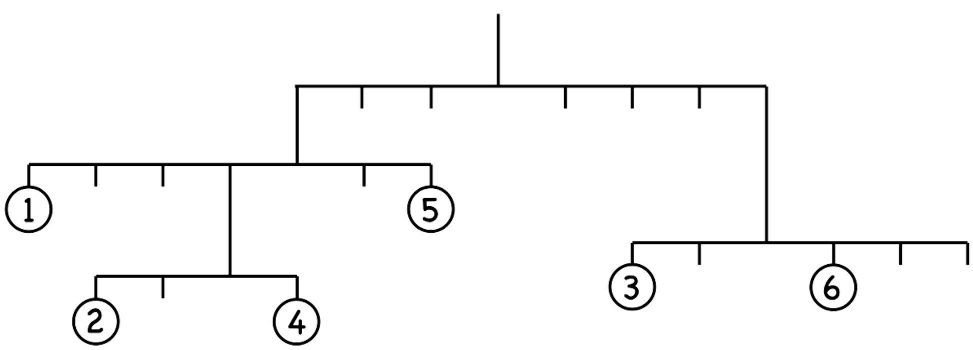
Start with the simplest mobiles, which are just levers in the air. Here is a solution for putting the weights from 1 to 4 on this mobile to balance it. This works as a lever with the fulcrum at the hanging point. For this mobile we have 2 x 4 + 1 x 2 = 4 x 1 + 3 x 2.



If there is more than one level to the mobile, then each individual arm on each level must balance as a lever. For this next mobile, the two bottom arms balance because 1 x 3 = 3 x 1 and 4 x 1 = 2 x 2. For the next level up, you just add up the weights below it. For example, the weight on the left side is 1 + 3 = 4 – as far as the next level up is concerned, it does not matter where on that bottom arm the weights are located. So, for the next level up, (1 + 3) x 3 = (4 + 2) x 2 = 12, so the top level balances as well.



Have fun making mobile puzzles for each other. Here is a last one to play with using each of the numbers from 1 to 6. Don't worry about being fancy and using each number once. Any completed puzzle will be fun. Checking the levels we have: 2 x 2 = 4 x 1; 1 x 4 + (2 + 4) x 1 = 5 x 2; 3 x 2 = 6 x 1; and (1 + 2 + 4 + 5) x 3 = (3 + 6) x 4.



Chapter 5 – Divide Up the Box

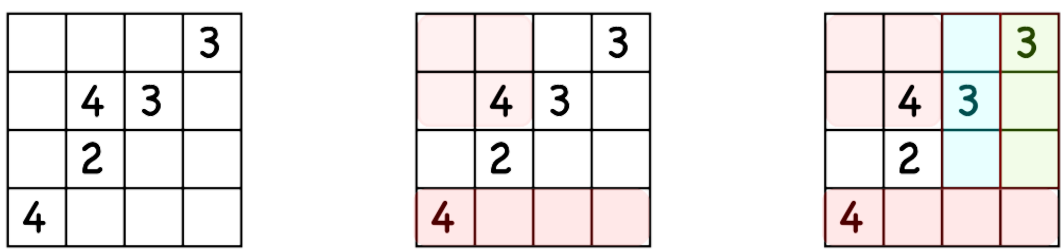
— Introduction —

A rectangle, 4 by 4 or larger, with numbers in some of its squares, is to be divided into smaller rectangles. Each number must end up in a separate rectangle whose area is that number.

For adults, constructing these puzzles is simple enough. Take a rectangle, divide its interior into rectangles, put numbers for the areas inside each interior rectangle, and then remove any sign of the interior rectangles. The only tricky part is putting numbers in places that make the puzzle reasonably easy to solve - you can always give hints as needed if your puzzle ends up being too hard.

— Solving Strategies —

Here are some general strategies that can simplify solving these puzzles. Do your best to let your child discover these rules as they play with the puzzles. Make a list together of the rules they come up with.



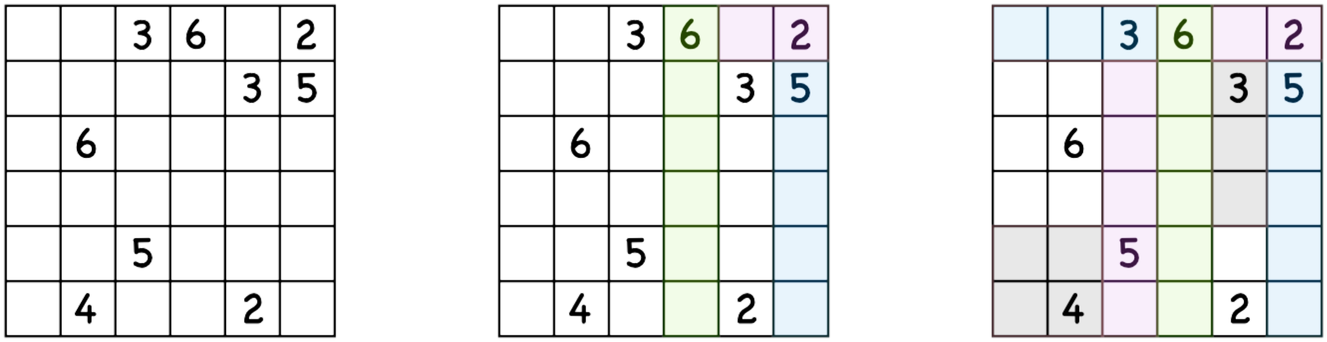
1) Look at numbers with only one or two options for their rectangles.

Both 4’s are highly constrained. Each 4 can only be inside a 1 by 4 or a 2 by 2 rectangle. The upper 4 is hemmed in, so it cannot be inside a 1 by 4. So, there must be a 2 by 2 rectangle in the upper left corner. That leaves the lower 4 with only the possibility that its rectangle is 1 by 4 and goes along the bottom side.

2) Look at prime numbers - they must be inside a 1 by n rectangle.

The 3’s in the puzzle above must be contained in a 1 by 3 rectangle. The 3 in the upper right corner can only be part of a 1 by 3 rectangle going along the top edge or along the right side. The upper left 2 by 2 square being blocked off for the 4 makes it impossible to have a 1 by 3 along the top edge.

The 1 by 4 along the bottom forces the 1 by 3 for the lower of the two 3’s to be the higher of the two vertical possibilities.



3) Numbers close to the maximum dimension often have few options.

Look at the 6’s and 5’s in this next puzzle. The uppermost 6 needs lots of room, and the only way there is enough room for it is vertically straight down, using up the entire column. The other 6 cannot be 1 x 6 because the row was cut off by the other 6’s column. So, the lower 6 must be a 2 x 3, which is not quite determined yet.

As another example, if there had been an 8 in this puzzle, 1 by 8 would not have fit, so it would have to be part of a 2 by 4 rectangle.

4) Squares that are boxed in have few options.

The uppermost 5 is boxed in, so it’s only choice is to be in a 5-box column. The other 5, because it is also a prime, must go vertically or horizontally. It is cut off horizontally by the column for the 6, so it must go vertically up to right below the 3.

5) Corners are often highly constrained.

The 2 in the upper right corner must go horizontally, so it is easy to fill in.

Chapter 5 – Letter Substitution Puzzles

— Introduction —

Once your child becomes comfortable with the Missing Number puzzles from a few pages earlier in this chapter, they can start playing with these puzzles. In these, one or more of the digits are replaced by letters. The three rules for letters are:

- A given letter is always the same digit
- The leftmost digit of a number is never 0
- Different letters must be different digits

Create these puzzles by taking an addition or subtraction problem and replacing one or more of the digits. The puzzles can also be created to make interesting problem-solving challenges for your child. Note that the values of the letters do not carry over from puzzle to puzzle.

— Examples —

This first example illustrates how you can take a standard addition or subtraction problem and make a letter substitution puzzle out of it. The first version replaced all the 6's with A's, and the second version went on to replace the 2's with B's.

$$\begin{array}{r} 23 \\ +46 \\ \hline 69 \end{array} \dashrightarrow \begin{array}{r} 23 \\ +4\mathrm{A} \\ \hline \mathrm{A}9 \end{array} \quad \begin{array}{r} \mathrm{B}3 \\ +4\mathrm{A} \\ \hline \mathrm{A}9 \end{array}$$

The rest of these examples are carefully constructed to allow solving using properties of the particular situation. One property to note is that when you add two numbers, the carry into the next column is always either 0 or 1. So, for example, in the problem A + A = C4, C must be 1 because it is not allowed to be 0.

$$\begin{array}{r} \mathrm{B} \\ +8 \\ \hline \mathrm{C} \end{array} \quad \begin{array}{r} \mathrm{B} \\ +\mathrm{B} \\ \hline 8 \end{array} \quad \begin{array}{r} \mathrm{A} \\ +\mathrm{A} \\ \hline \mathrm{C}4 \end{array} \quad \begin{array}{r} \mathrm{A} \\ +2 \\ \hline \mathrm{BC} \end{array} \quad \begin{array}{r} \mathrm{D} \\ +2 \\ \hline \mathrm{EE} \end{array} \quad \begin{array}{r} \mathrm{A} \\ +\mathrm{B} \\ \hline \mathrm{AC} \end{array}$$

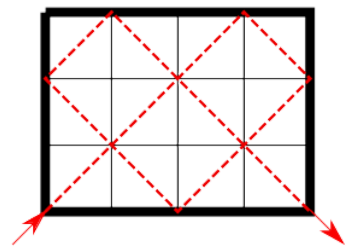
$$\begin{array}{r} \mathrm{A} \\ \mathrm{A} \\ +\mathrm{A} \\ \hline \mathrm{B}2 \end{array} \quad \begin{array}{r} \mathrm{C} \\ \mathrm{C} \\ +\mathrm{C} \\ \hline \mathrm{D}4 \end{array} \quad \begin{array}{r} \mathrm{A} \\ \mathrm{A} \\ +7 \\ \hline \mathrm{B} \end{array} \quad \begin{array}{r} \mathrm{A} \\ \mathrm{A} \\ +\mathrm{B} \\ \hline \mathrm{B}0 \end{array} \quad \begin{array}{r} \mathrm{A} \\ +\mathrm{BB} \\ \hline \mathrm{A}7 \end{array} \quad \begin{array}{r} \mathrm{B} \\ +\mathrm{AB} \\ \hline \mathrm{BA} \end{array}$$

$$\begin{array}{r} \mathrm{BA} \\ +\mathrm{BB} \\ \hline \mathrm{CAB} \end{array} \quad \begin{array}{r} \mathrm{AD} \\ +\mathrm{BD} \\ \hline \mathrm{BCC} \end{array} \quad \begin{array}{r} \mathrm{AA} \\ +\mathrm{BA} \\ \hline \mathrm{BBC} \end{array} \quad \begin{array}{r} \mathrm{AA} \\ +\mathrm{CB} \\ \hline \mathrm{BBC} \end{array} \quad \begin{array}{r} \mathrm{AA} \\ +\mathrm{AB} \\ \hline \mathrm{CAC} \end{array} \quad \begin{array}{r} \mathrm{AA} \\ +\mathrm{AA} \\ \hline \mathrm{BBC} \end{array}$$

Chapter 5 – Playing with Shapes

— Bouncing Billiard Ball – Introduction —

Imagine a billiard table that has a pocket in each corner. When a ball bounces off the side of the table, it bounces away at the same angle it came in at. If we shoot a ball at a 45 degree angle from the lower left corner, where will it end up? The answer depends on the size of the table. Pictured at right is what happens on a 3 by 4 table.



Give your child a drawing of a table and challenge your child to predict which corner will be hit first and how many bounces it will take before getting to that corner.

— Bouncing Billiard Ball – Analysis —

Start by letting your child just play around with this and don't be in a hurry discovering results. As you will see, this problem involves some sophisticated ideas for a young person. As needed, ask a question or two to give their thinking a little more structure. You know what's coming – look at simpler tables first to look for patterns - when this idea becomes automatic for your child, this will serve them well for the rest of their lives!

The simplest tables are 1 by n, and they are easy to understand. Playing with a few values of n, the pattern emerges quickly. It is easy to undervalue a simple result like this; however, any completely understood result is to be celebrated, and this result will lead to others.

<u>Result</u>: 1 by n table: The ball will take n-1 bounces. The ball will end up in the bottom right corner if n is even and in the upper right corner if n is odd.

The next simplest tables are 2 by n. The patterns here are a little more involved. Good record keeping can make a big difference in something like this. An observant experimenter will notice that a 2 by 4 table behaves just like a 1 by 2 table, and a 2 by 6 table just like a 1 by 3. This quickly generalizes to the next result.

<u>Result</u>: A 2 by 2xn table behaves just like a 1 by n table.

Why is this? What is going on? This is a mathematical process to instill in your child – look for patterns and then seek to understand them, and with that new understanding extend your earlier results.

What is going on is that the bounces on a table do not change if you enlarge both dimensions by the same factor. When that is done, the table is bigger but the geometry is the same. In geometry terms, the two tables are said to be "similar."

<u>Result</u>: A kxm by kxn table behaves exactly like an m by n table.

We have gotten here in little steps, but this is a BIG result. It means we can start our analysis on any table by first removing any common factor.

Resuming where we left off for 2 by n tables. We understand what happens when n is even, but what happens when n is odd? What happens for 2 by n for n = 1, 3, 5, 7, and so on? The pattern quickly becomes easy to see.

<u>Result</u>: When n is odd, a 2 by n table has n bounces and ends up in the upper left hand corner.

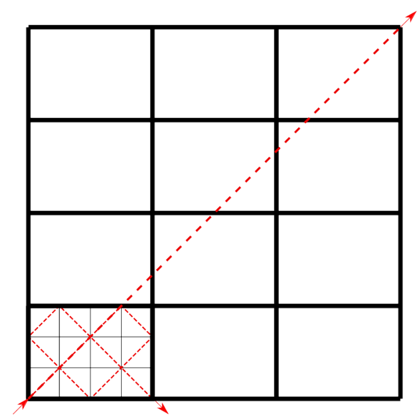
Lots of progress is being made. Playing with more examples leads to some more patterns.

<u>Result</u>: If n is not a multiple of 3, a 3 by n table has n+1 bounces and ends in the upper right corner if n has a remainder of 1 when divided by 3, and in the lower right corner if n has a remainder of 2 when divided by 3. If n is odd, a 4 by n table has n + 2 bounces and ends in the upper left corner. If n is not a multiple of 5, a 5 by n table has n+3 bounces and ends up in the upper right corner when n is odd and lower right corner when n is even.

At this point we are tempted to look over the data, see some patterns, and make some conjectures.

<u>Conjecture</u>: Assume k and n have no factors in common. Then a k by n table will have k + n - 2 bounces. It will end in the upper left corner if k is even. It will end in the upper right corner if k is odd and n is odd, and in the lower right corner if k is odd and n is even.

Wow - if this conjecture is true, we have completely solved this problem! You know what's coming… Let's see if we can explain why this conjecture should be true (or find out that it is false).



Although there are other ways to understand this situation, as sometimes happens, what makes this problem much easier to understand is a new idea. It might not occur to you, but once you see it you will probably be amazed. The idea is to unfold the table so that the ball can go in a straight line! Here is what happens if we unfold the original 3 by 4 table and make the path of the ball into a straight line.

Seeing that the conjecture is true is a lot easier now. The bounces correspond to crossing lines - there are (k - 1) of them to cross in one direction and (n - 1) of them to cross in the other direction, so together that makes a total of (k - 1) + (n - 1) = k + n - 2 lines to cross. Seeing which corner it ends up in is a matter of keeping track of how things unfold. We're all done now with quite an interesting journey.

— Filling Regions With Shapes – Introduction —

Suppose you have an 8 by 8 chessboard and you have a collection of 1 by 2 tiles. Finding a way to exactly cover the chessboard with 32 of these 1 by 2 tiles is simple enough.

Let's start removing some squares from the chessboard and see what happens. If you remove one corner of the chessboard, you know immediately that you can no longer cover the chessboard with tiles because the tiles will always cover an even number of squares, and there are now 63 squares to cover. Okay, remove two corners to make an even number of remaining squares - can you cover it now? The answer depends on which two corners you remove. Why? What if you no longer restrict yourself to removing corners, what will happen then?

— Filling Regions With Shapes – Analysis —

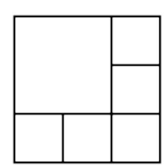
Let your child play with this before revealing the coloring idea. If they play around with small boards, they may discover the rule on their own, and that is always better.

An observation that helps a lot with this question is to use the coloring of the chessboard squares. If you take the 1 by 2 tiles and color one square white and the other black, you will see an interesting thing occur. Every tile must cover a square of each color. Not only will k tiles cover 2xk squares, but they will cover k white squares and k black squares - the same number of squares of each color. Using this idea, it becomes obvious that if you remove more squares of one color than another, it will be impossible to cover the board.

If your child is enjoying these questions, start branching out to using other shapes to fill the board. Play around with filling it with 1 by 3 tiles or with 3 squares in an L shape. What patterns and rules do you discover with these? What other shapes might be interesting to play with?

— Filling Squares With Squares – Introduction —

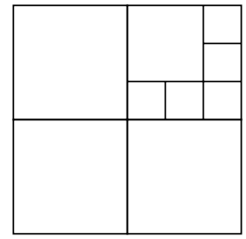
In which ways can you fill a square with other squares, where the other squares need not all be the same size? However, the lengths cannot be totally random numbers – the side length of each square must be some whole number multiple of a fixed length. The question to investigate is: What are all the numbers of squares that are possible? Also, if you know a number is possible, is there an easy way to describe how to do it?



Let your child play with it over many days and don't be in a hurry to get to the answer. There are many different ways to come up with ideas for this investigation, so be flexible and work with your child's ideas. Here is a diagram showing how 6 is possible.

Coming up with some quick examples is always a good idea. Breaking the big square into squares of equal size as an easy start. From that you know that the square numbers (1, 4, 9, 16, 25, …) all work.

Working off the 6 square example, we can use one large square of any size and put 1 by 1 squares on two of its sides. Doing that for ever larger squares (1 by 1, 2 by 2, 3 by 3, …) we get 1 + 3 = 4, 1 + 5 = 6 (as pictured), 1 + 7 = 8, 1 + 9 = 10, and so on. So, all even numbers starting with 4 can be done this way.



A powerful idea that wraps this up quickly is to see that we can take a diagram that works, and replace one of its squares by another diagram that works. So for example, if you take a simple 2 by 2 filled in with 4 1 by 1 squares, and you replace one of those 1 by 1 squares with the 6-square example, you get the diagram shown at right with 9 squares.

Because one square is getting replaced by an n-square diagram, the net change in the number of squares is to add n-1 of them. That means that we can take one number that works, and add multiples of one less than it to any other number that works. In particular, we can add multiples of 4 -1 = 3 to any other number that works - the easy ones to add 3 to are all the even numbers starting with 4.

Putting that all together says that the numbers 1, 4, 6, 7, 8, 9, … all work, and it is easy to see at least one simple way to construct them. It is also easy to convince yourself that 2, 3, and 5 are impossible.

If your child enjoys exploring that question, explore variations on this theme. Suppose you only allow squares of certain sizes - such as 1 by 1, 2 by 2, and 3 by 3. Or perhaps only allow 2 by 2 and 3 by 3. See which questions lead to interesting results and which ones are not so interesting.

Another direction to look at is filling other figures with figures that have the same shape. For example, ask the same question for regular triangles (triangles with all their sides the same length). Some figures are interesting to investigate in this way, and some are not interesting at all - which ones?

Chapter 5 – Product Game

— Introduction —

Use a shared piece of paper filled out as follows:

The first player moves a token onto any number from 1 to 9 in the 1-9 squares on the bottom row. The second player puts another token on one of the 1-9 squares on the bottom row and claims the product in the 6 by 6 grid. From then on, each player chooses to move one of the two tokens and claim the product (if they can). The first player to claim 3 squares in a row wins. Mix up the product numbers in the 6 by 6 grid to give your child better practice identifying the products.

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

1	2	3	4	5	6	7	8	9

These playing boards can be made as large as you like, though they do get to be quite large pretty quickly. Here are a few larger boards with the corresponding larger number ranges beneath them.

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
50	54	56	60	63	64
70	72	80	81	90	100

1	2	3	4	5	6	7	8	9	10

★	1	2	3	4	5	6
7	8	9	10	11	12	14
15	16	18	20	★	21	22
24	25	27	28	30	32	33
35	36	40	42	44	45	48
49	★	50	54	55	56	60
63	64	66	70	72	77	80
81	88	90	99	100	110	121

1	2	3	4	5	6	7	8	9	10	11

★	1	2	3	4	5	6	7
8	9	10	11	★	12	14	15
16	18	20	21	22	24	25	27
★	28	30	32	33	35	36	40
42	44	45	48	★	49	50	54
55	56	60	63	64	66	70	72
★	77	80	81	84	88	90	96
99	100	108	110	120	121	132	144

1	2	3	4	5	6	7	8	9	10	11	12

The squares with red stars are "free" squares and may be used by either side as needed.

Chapter 5 – Limited Calculators

— Introduction —

Suppose you have a calculator that is badly broken and you are challenged to produce some result on the calculator. You can come up with a wide variety of scenarios that can provide interesting challenges with a quick puzzle description. This activity is easy to play orally whenever you have a spare moment. Here are some examples to get you started.

Although there are some moments where deeper mathematics is going on in these questions, mostly these are problems entirely for the fun of playing around with them.

1a) Suppose you had a calculator with +, -, x, and /, but only one working number key, the 4. Could you get the result 21? If so, what is the fewest number of steps you would need?

4 + 4 + 4 + 4 + 4 + 4/4 = 21 is one way, but there are many other ways to do it. Another is 4 x (4 + 4/4) + 4/4. The goal is to play around and enjoy the exploration.

1b) Suppose you could use 4 at most four times - which numbers could you produce? Suppose you had to use the 4 exactly four times.

As a child's math resources increase, the four 4's problem is a fun puzzle. At this point, your child's choices are quite limited, but it is still a lot of fun to play around with. It will be particularly difficult to do many of the numbers without dividing or using decimals. Don't be concerned with coming up with all the numbers in order - just come up with as many different numbers as possible.

Here are a few examples just to get you started.

1 = (4/4) x (4/4) = 44 / 44

2 = 4 / ((4 + 4) / 4)

3 = (4 + 4 + 4) / 4

4 = (4 - 4) x 4 + 4

6 = 4 + (4 + 4) / 4

7 = 44 / 4 - 4

8 = (4 + 4) x (4/4) = 4 + 4 + 4 - 4

32 = 4 x 4 + 4 x 4

1c) Play around with having other single numbers and creating other results.

2a) Suppose your calculator could only add 4 or 7. Which numbers could you produce?

This is the result we have seen several times by now. Starting at (4 - 1) x (7 - 1), you can achieve all numbers by adding multiples of 4 and 7. 18 = 2 x 7 + 4, 19 = 3 x 4 + 7, 20 = 5 x 4, 21 = 3 x 7, and so on.

2b) Suppose it had 4 or 7, but it could add and subtract. Which numbers could you produce?

You can produce all numbers in this way.

2c) Replace 4 and 7 with other pairs of numbers. What happens for these pairs?

In Number Theory, this is called Bezout's Theorem. The result says that by combining multiples of two numbers you can produce any multiple of the greatest common divisor of the two numbers.

3) Suppose you only had a 1 key and could only add or double. For example, 2 x (2 x 1) + 1 is 5. What other numbers can you create?

This is a question about binary numbers in disguise. It is not important for your child to realize this or understand it, it is just for playing with. Any number can be written in binary, so all numbers can be achieved by combining doubling with adding 1. For example, 21 is 16 + 4 + 1. So, 21 = 2 x (2 x (2 x (2 x 1) + 0) + 1) + 0) + 1.

Chapter 5 – Double or Nothing

— Introduction —

Players start the game by secretly picking 5 distinct numbers larger than 20 and not bigger than 120. After they are selected, they are written where all can see them. Using Number Cards or some other device, a random number from 1 to 20 is created. That number is repeatedly doubled until either someone's number is hit for the first time or the number becomes bigger than 120. The first player to have all five numbers hit is the winner.

— Analysis —

The question is: What are the best five numbers to pick? Here are some ideas to think about.

<u>Rule</u>: Always pick a number that is a power of 2 times a number from 1 to 20.

If you pick a number such as 23 or 46, they can never be hit and you are guaranteed to lose.

<u>Rule</u>: Never pick a number that is twice another number you could have picked but didn't.

If you pick 44, why not pick 22 instead? If the other person picks 22, you will miss a round.

<u>Further analysis</u>: The numbers from 1 to 20 are equally likely to be picked. However, because 9 leads to 18, 18 is twice as likely as a starting point than say 11 is. If you combine the ways to get different starts, the starting points have the following probabilities:

11 - 1/20 (from 11)

12 - 3/20 (from 3, 6, and 12)

13 - 1/20 (from 13)

14 - 2/20 (from 7 and 14)

15 - 1/20 (from 15)

16 - 5/20 (from 1, 2, 4, 8, and 16)

17 - 1/20 (from 17)

18 - 2/20 (from 9 and 18)

19 - 1/20 (from 19)

20 - 3/20 (from 5, 10, and 20)

Clearly the best numbers to use are multiples of 16, 12, and 20. A simple strategy is to use the five numbers: 32, 64, 24, 48, and 40. These numbers will not always win, but they should do very well for you over time.