Early Family Math

Math Learning Steps

This is a collection of brief descriptions of mathematical steps your child will progress through as their mathematical world grows and develops. Its purpose is to present each learning step, as well as to provide a sense of how a step builds on earlier steps and lays the foundation for future steps. It does not have the completeness and depth needed to train a math teacher in a school.

We separate early math learning into the following Stages. The ages associated with these Stages are meant to be approximate, and they will vary widely for different children in different circumstances.

- Stage 1: I Can Hear You! – Ages 0 to 3
- Stage 2: I Can Count to 5! – Ages 2 to 5
- Stage 3: I Can Count to 10! – Ages 3 to 6
- Stage 4: I Can Count to 20! – Ages 4 to 7
- Stage 5: I Can Count to 100! – Ages 5 to 8

The name of each Stage is chosen to suggest a simple-to-identify skill level requirement for starting that Stage. This simple structure will not fit every child perfectly, but it does provide useful guidance as to where you should start.

In the following pages, we divide each of these Stages into a sequence of ten learning steps.

Stage 1: I Can Hear You! – Ages 0 to 3

This Stage starts as early as 4 to 6 months. Exposing your child to all kinds of experiences is so important. Start talking with your child even though they show no outward signs of understanding you. Make a habit to point at and describe all sorts of things. As your child starts reacting to your words, start mixing in questions for your child to respond to.

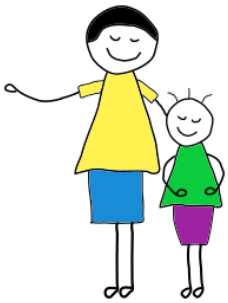
When your child begins to understand that objects have properties, start pointing out properties that make things similar or different. These properties are also useful for playing with patterns. Finally, start describing shapes in terms of their many properties.

The math learning steps for this Stage are:

1. Math Talk – Early and Often
   Constantly pointing at and describing things is very important for a child's development. Start doing this long before they seem to understand.
2. Math Talk – At Home
   There are many mathematical things in your home to talk about.
3. Math Talk – Out and About
   Talk about math in a store, in the park, and in many other places. Math is all around us!
4. Math Talk – Describe, Compare
   Use describing, comparison, and spatial relationship words with your child - this is math!
5. Math Talk – Counting
   Whenever possible, count things for your child.
6. Math Talk – Point, Describe, and Ask
   Begin asking questions that they can respond nonverbally to by pointing or actions.
7. Object properties
   Your descriptions and play should start to involve lots of properties of objects.
8. Same and Different
   Discuss properties that make things the same or different
9. Patterns
   Start playing with identifying, creating and extending patterns.
10. Basic shapes I
    Introduce geometric ideas and objects, and the names of simple shapes.

1.1 MATH TALK – EARLY AND OFTEN

Exposure: During these early years, it is all about exposure! Your child is being exposed to a wide array of experiences and is discovering patterns in everything they sense. As part of exposing your child to the world, expose them to math words and ideas. Let them see how enjoyable it can be to play with math together.



Start early: Start this even before your child seems to understand what you're saying. Your child is a sponge who is getting more from your words than you realize.

Point and describe: Point at things your child interacts with and describe them with words involving numbers, shapes, and colors. If you are dealing with a small set of things, count them out loud to your child.

Many facets of Math Talk

There is much more math to talk about than just numbers.

- *Describe things. Talk about sizes, colors, textures, shapes, softness, wetness, hotness, brightness, and more. Naming and describing properties is essential for comparing them and discovering patterns.*
- *Use comparison words. Bigger, smaller, tallest, widest, more, less, same, ...*
- *Use position words. Over, under, between, near, far, above, ...*
- *Talk about patterns and sequences in space and time. Refer to the order of things as first, second, third, and last. Talk about what just happened, what is about to happen, and things happening today. Talk about patterns in designs you see.*
- *Count things out loud and say numbers to refer to quantities.*
- *Use measurement words. Use words for length, area, weight, and volume whenever you are describing sizes.*

This is all math: These different ways of describing things and their relationships is talking about math! Building up this vocabulary and concepts will help your child develop mathematically. It will also give a big boost to helping your child read and talk about the world.

1.2 MATH TALK – AT HOME

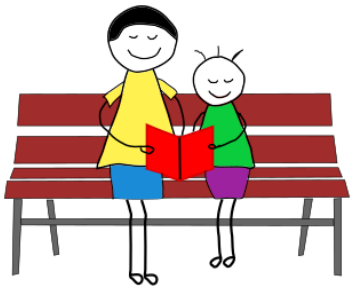
As you go about your household routines and activities together, there are many mathematical conversations to have with your child.

Putting things away: Talk about which things belong together. Do things that are the same shape go together? Is there a special place for round things or triangular things?

Clothes: If you are sorting clothes that are about to be washed or have just been washed, talk about colors, shapes, and sizes. When picking up clothes or putting them away, talk about why some clothes go in one place and others go in other places.

Going to sleep and getting up: These times lend themselves to discussing doing things in order, and practicing words like first, second, third, last, and next.

Discuss as you read: Storytime is a wonderful chance to do math with your child in a cozy setting. Talk about the characters and the things in the pictures. If there is a big yellow sun, point to the sun and say: "The sun is round and it is yellow. The wall of this room is also yellow. Point to something round in this room."



As your child gets older, count together by pointing, such as the yellow flower petals in a picture, or ask your child to point to things you describe.

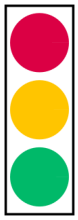
Food: Whether putting food away, cooking food, or setting things out for mealtime, there are many opportunities for math. Different kinds of food should be put away in particular places – this is a good time for relationship words like inside, under, and over.

Cooking involves measuring quantities, talking about lengths of time, and describing the desired result for the food. Setting things out for mealtime involves setting out the appropriate number of things so that each person gets what they need.

Playing with objects: Compare objects when constructing things for play or other uses. Which one is taller? Can you make one thing taller, wider, bigger, or the same as the other? Describe and compare the sizes, numbers, and colors of things you have or that are in pictures.

1.3 MATH TALK – OUT AND ABOUT

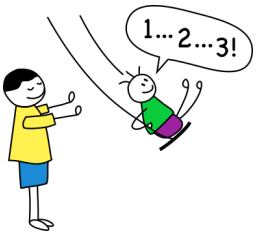
Shapes: You might see a circle in a design in a building and ask your child to point out other circles they see, such as the circles in a traffic light. Traffic signs and shop signs provide a great supply of shapes you can describe and name. There is no end to the shapes, colors, and counting that you can find and talk about once you make a habit of looking for them.



Traveling: There are many mathematical things to talk about as you travel. If you see a somewhat unusual red car, you can point that out and count together other red cars that are like it. Ask about bigger, smaller, thinner, and wider things such as buildings, windows, trees, and people. Which things are closer than others, and which things are farther away?

Counting in a store: Talk about how many apples you need, and count them out as you pick them out. Count the people in line in front of you, and compare that to the length of the other lines.

Point out the shapes of fruit or pictures on food boxes. Talk about how some things come in boxes, and other things come in round bottles. You might need something on a high shelf, or something on a low shelf. There is so much to describe and compare!



In a park: Count the children, the number of structures or trees, or anything else. Comment about where there is more of one thing than another.

On the swings: Pushing your child on something that swings or sways back and forth is a perfect opportunity to count with your child. With each push, count "1, 2, 3, 4, 5." After your child starts learning how to count to 5, counting down from 5 is also a good idea. Start or end at 0 sometimes.

Point out the circles, curves, straight lines, triangles, and rectangles in the park. Talk about how some things are over, under, between, or on top of other things.

1.4 MATH TALK – DESCRIBE, COMPARE

There is important math beyond numbers: Describing and comparing things is an important part of helping your child learn early mathematics. When children do mathematics, they use ideas about properties of objects to help them work with those objects, such as in grouping them or finding patterns with them. These skills will also help your child when it comes time to start learning to read.

Point, describe, and compare: Wherever you are, get in the habit of pointing at things that catch your or your child's attention, and then describing them. Take the opportunity to compare those things to other things to help the descriptions be more meaningful. Talk about how two things are the same or how they are different.

It is never too early: From the earliest ages, your child is learning from everything they see, hear, taste, touch, and experience. Add to those experiences with your Math Talk. They are benefiting from your words before they show any reaction to them. Eventually, they will put together the experiences from your words and start showing you that they understand them.

Expose, but do not rush: Don't confuse early exposure with teaching. A child will put together the patterns from their experiences as they are developmentally ready. For example, you can't teach your child to count to 5 by explaining it to them – you can only expose them to your counting over and over again until it starts making sense to them. Never be impatient or push them to understand it more quickly – they have a lot to learn and they will naturally want to make sense of it all.

Searching game: Make a game of searching for objects around where you are. Use the game to practice concepts that your child is learning, such as color, size (large, medium, small), weight (heavy, light), quantity, and relationship (inside, on top of, below).

One of you says to the other that they see something that is round on top of something that is brown. The other person tries to discover what it is. If they have trouble finding it, more clues are given.

1.5 MATH TALK – COUNTING

Counting, Numbers, and Quantities: Counting and numbers are what comes to mind when most people think of early math, and they are the easiest to relate to and understand. Counting is also easy to verbalize in front of your child. There are several things to work on at the same time, so it can be more complicated for your child than what you might think.

- Counting forward and backward, including 0 sometimes
- Learning the numbers
- Learning quantities

Repeating the sequence: At first, all your counting in front of your child will lead to your child starting to repeat the numbers in sequence. Don't be surprised if they leave out some of the numbers or they say the numbers in the wrong order. If they make these errors, don't make a big deal out of it; simply say the count correctly and move on. This is all part of the process and they will eventually learn the numbers in the correct order.

Count down sometimes: Counting down will help your child understand the sequence. This helps keep it from being a meaningless sequence of sounds they make that makes adults happy. You can do this almost any time you would have counted up. For example, if you have three apples you want to remove, count down from three after you take away each apple.

Include 0 sometimes: Start your counting at 0 sometimes to help 0 become an accepted number and quantity. You can also count down to 0. Counting down to 0 is great for events that are about to happen, say in ten seconds. You can count down to 0 and then say "Blastoff" or something similar.

Understanding quantities: It is completely intuitive to an adult that if you count a collection of things, say four pebbles, that when you count "1, 2, 3, 4" the last number you say is the size of the quantity of things you have. Your child is learning several things about counting and quantities. They are gaining an understanding of quantities. They are learning that when they count something they do a 1-to-1 correspondence between the things and the numbers they're saying. They are learning that it doesn't matter in which order you count things. And finally, they are learning the "last number is the size" rule. Adults take these things for granted, but it is a great deal for a child to learn. Be patient and remember that there is no rush.

Count everything: There are so many things to count. Count chairs, steps to go a short distance, food items, people in line, people in a group, furniture around a table or in a room, arms, legs, and who knows what else. Whenever you find yourself mentally counting something, count it out loud in front of your child, and be sure to count down sometimes and to include 0 sometimes.

1.6 MATH TALK – POINT, DESCRIBE, ASK

Responding: Once your child is able to respond to what you say, that will allow you to start asking questions. Your "Point and Describe" Math Talk wlll now become Point, Describe, and Ask. Even before your child can say words, they will start to respond nonverbally to you as they begin to make sense of your words.

Ask questions: Use this new level of communication to make it clearer which ideas your child understands. "Where is the bird?" or "Where is the ball? makes it clear if your child knows what a bird or ball is. Similarly, "Point to the tree." or "Point to a car." works for those two concepts.

Handling mistakes: Ask all kinds of questions. Ask which of two things is bigger or smaller. Ask where you are walking to. Ask where something belongs. All these questions are opportunities for your child to express their understanding and for you to clear up any misconceptions. If your child points to the wrong thing or picks up the wrong thing, simply point out the right thing and don't try to explain their mistake to them.

Do this while reading: Now you can add asking questions to your pointing and describing as you read a story. Point to pictures in the story and ask the same questions you would ask about the things around you.

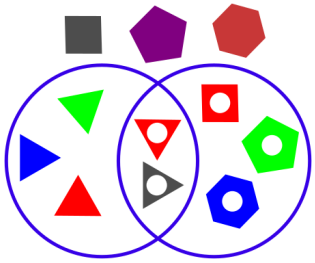
Simple riddles: Play games with your child's new skill and have fun with it. Have fun unraveling puzzles together such as: "There is something red under something blue. Where is it?"

1.7 OBJECT PROPERTIES

Your child responds!: All the pointing, describing, and asking you have been doing with your child has established that things have properties that can be discussed and reasoned with. You have been building up a vocabulary of words and concepts for describing things. It is time to start making more use of them.

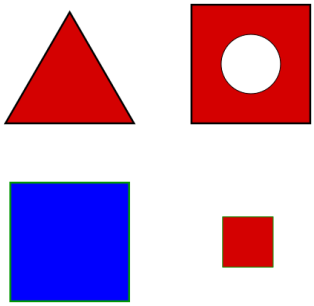
Ask for things with a specific property: Practice using properties by asking your child to bring you something with that property. You could ask "Please bring me something that is red." for example. As your child gets better at this, make the requests more complicated by combining more than one property – "Find a round wooden thing."

Grouping things with a property: Practice grouping things with the same property. If your child has a collection of objects, ask to have all the round things put to one side.



A circle for each property: Make this more visible by drawing a big circle and having all the things that have a particular property put in that circle. For example, you could have all the things with a hole in them put in the circle. As this becomes easy for your child, use two circles that overlap – one circle could be for triangles, the other for things with holes, and the common area to the two circles would be for triangles with holes.

Which one doesn't belong?: A fun activity for practicing with properties is to show your child a small set of objects and ask which of them doesn't belong. Challenge your child to identify the object that is not like the others and to explain why. Accept any reason that makes sense; your child may have an unusual reason.



For example, you could have pictures of some animals. Perhaps only one of them can fly. Maybe only one of them has two legs. This activity can provide fun challenges that let your child do some creative thinking with new concepts.

1.8 SAME AND DIFFERENT

Comparing: Help your child understand properties better by comparing and contrasting them.

Examples: For example, talk about the size or age of a child and an adult – one is younger and the other older, and one is shorter and the other is taller. Or you could talk about a bird and a dog – one can fly and has feathers and the other has fur and cannot fly.

Same and different: Make a playful activity out of this by showing your child two objects and asking how they are the same and how they are different. Be prepared for surprising ideas and do include silly suggestions along with the more serious ones.

More examples: If you hand your child a spoon and fork, there are many things your child might say. They are the same because you eat with both of them. They are also the same because you hold both of them, they are about the same size, or they are made of the same material. They are different because one is smooth and somewhat round, while the other is pointed.

1.9 PATTERNS

Patterns are everywhere! Recognizing, describing, and creating patterns is central to playing with mathematics.

Here are some characteristics that can be used by themselves or mixed together to create patterns:

- Movement patterns: stepping, jumping, waving, nodding
- Sound patterns: clapping, knee slapping, tongue clicking, stamping
- Loudness patterns: soft, medium, loud
- Visual patterns: color, shape, size

Discover patterns: Challenge each other to find repeating patterns wherever you are. You might notice a repeating tile work in a floor, wall, or ceiling. The brick work of a building might make an interesting pattern. Plants may be planted in an organized pattern in a field. The side of a pineapple or pine cone may have a spiral pattern. Something may be producing sounds in a repeating pattern.

Game: Repeat patterns: Two or more of you can challenge each other to repeat and extend each other's patterns. This can be done in many ways. The simplest is for one person to create a pattern of sounds and movements and have all the others repeat it.

Add difficulty to this by having the original person add one more item to the end of the pattern every time the pattern goes once around the group. Alternatively, each person can take the pattern that comes to them and one more item at the end of it.

Secret handshakes or knocks: Use patterns as an agreed upon way for being allowed to enter some place such as a room. This might be a series of fist bumps and other type of handshakes. Or it might be knocking and stamping that causes a series of sounds.

Sequence of drawings: For older children, create puzzles by drawing a pattern of shapes. One person establishes a pattern and then leaves gaps in the repeating sequence for the other to fill in.

1.10 BASIC SHAPES I

The world of shapes: There are many possibilities that open up as your child's understanding of properties grows. For example, they can now understand geometric shapes and talk about them!

Counting sides: Distinguishing between a triangle, square, rectangle, hexagon, and octagon will involve the idea of "sides" and being able to count those sides. Your child's growing understanding of quantities will make this counting possible. Also, as your child begins to automatically identify these shapes, that will help them to deepen their understand of the quantities for 3, 4, 6, and 8.

Basic shapes: For now, stay with basic shapes and build confidence and fluency with those shapes. Of course, you can always mix in any additional shapes, such as stars, that your child happens to enjoy.

Here is a list of words for flat shapes:

- Circle
- Triangle
- Rectangle (looks like a piece of paper)
- Square (rectangle with equal sides)
- Hexagon (6 sides)
- Octagon (8 sides – stop sign)

This is a list of words for 3-dimensional shapes:

- Ball (Sphere)
- Cylinder (Round tube)
- Box (Cube)

Hands on: Use and explain these shape names as your child plays with objects that have these shapes. Give them lots of opportunities to see how these fit together or build on top of each other. Help your child identify these shapes as you see the shapes out in the world.

Stage 2: I Can Count to 5! – Ages 2 to 5

Your child has mastered counting to five and understands those numbers in terms of quantities. They also understand that objects have various properties and that those properties can be compared and contrasted.

During this Stage, your child will learn to count to ten and begin to understand adding and subtracting of small numbers. They will also become much better at understanding the size of small quantities of objects.

1. Counting up from 1 and 0
   This is the foundation of understanding quantities, and builds a basis for addition and subtraction.
2. Counting down to 1 and 0
   This solidifies the understanding of the number sequence and helps with subtraction.
3. Comparing small quantities
   Use quantities of objects to build understanding of relative sizes.
4. Counting On
   Counting On helps with an understanding of quantities, saves time for counting, and is important for adding.
5. Basic shapes II
   Use more sophisticated geometric ideas of shapes.
6. One more and one less
   Learning the next number and previous number are the first steps towards learning addition and subtraction.
7. Finger add and subtract to 5
   Small sums up to 5 can be handled effectively with one hand.
8. Finger add and subtract to 10
   Using your child's fingers is an effective way to do addition and subtraction to 10.
9. Skip counting by 2's
   This is a fun and fast way to count a group of objects.
10. Counting groups
    Explore different ways of counting groups of objects.

2.1 COUNTING UP FROM 1 AND 0

Count everything: Your child can already count to 5. You are now helping them to solidify that counting and as well as extend it to larger numbers. Count things in front of your child and also count with your child when your child wants to. Counting is the foundation of understanding quantities. It also builds a basis for addition and subtraction.

There are so many things to count. Count chairs, steps to go a short distance, food items, people in line, people in a group, furniture around a table or in a room, arms, legs, and who knows what else. Whenever you find yourself mentally counting something, count it with your child, and be sure to include 0 as a starting point sometimes.

Understanding quantities: It is completely intuitive for an adult that if you count a collection of things, say four pebbles, that when you count "1, 2, 3, 4," the last number you say is the size of the quantity of things you have. Your child is learning several things. They are gaining an understanding of quantities. They are learning that when you count something you do a 1-to-1 correspondence between the things and the numbers. They are learning that it doesn't matter in which order you count things. And finally they are learning the "last number is the size" rule. Adults take these things for granted, but it is a great deal for a child to learn. Be patient and remember that there is no rush.

Mistakes: There are many kinds of mistakes you should expect your child to make as they count. Your child may leave out numbers or they may leave out some of the numbers. Or they may get mixed up in doing 1-1 correspondence with a group of things they are counting. Your child will sort all this out with time. For now, when your child makes a mistake, simply count the things out in front of them correctly and move on to something else.

Understand the numbers: Don't be in such a hurry for your child to repeat the numbers 1 to 10 that they have no idea what they are saying. Take time with it and reinforce the connection of each number with its corresponding quantity.

Reading numbers: You may of course start introducing reading the numbers whenever you like. However, don't let the reading limit the counting. Learning to read numerals will typically take longer than learning to say the numbers and getting a sense of the quantities.

2.2 COUNTING DOWN TO 1 AND 0

It helps with meaning: Counting down is surprisingly effective at solidifying understanding of the number sequence. Many very young children who have learned to count from 1 to 10 have a difficult time when they first try to count from 10 down to 1. This counting in the opposite order forces them to think in a fresh way about how the numbers are ordered. This will be particularly true when your child starts counting to 100 and they start thinking about what happens as they go from one decade to the next – say between 69 and 70.

It helps with subtraction: Counting down is also very helpful with learning to subtract. A child who can count down will very quickly learn how to subtract 1 and 2. It is also helpful if counting down becomes automatic, so they can use their full attention on counting down the three steps when doing 9 subtract 3, for example.

Use when natural: There are many times when counting down is the natural thing to do. If there are ten seconds left on a timer, you can count down together with the counter. If you say that something can be done three more times, then you can count down from three together.

Include 0: It is usually natural to include 0 when counting down, and this is a good thing to do. If you are counting down the time remaining, when you reach 0 you have 0 seconds left. If you count down how many pieces of food your child can eat, when you reach 0 there will be no food left. It is helpful to make 0 a normal and expected quantity to work with.

2.3 COMPARING SMALL QUANTITIES

It takes time: All these beginning concepts and skills take lots of time for your child to master, and comparing quantities and getting a sense of the size of quantities is no different. Create many, many experiences that allow your child to touch, feel, and directly experience the sizes of quantities and how they compare.

Line them up: One simple way to compare the sizes of two quantities is to line them up in 1-to-1 correspondence next to each other. For example, if you play a card game and you want to see who won, you can line up the cards of the two players and see which line has extra cards.

Common misunderstanding: Be aware that a small child may think that two short sticks is the same amount as one long stick. This is perfectly understandable, but it is often not what the adult had in mind.

Give choices: Put your child in situations where they can choose between two groups of of the same things they really like. When they pick the bigger group, reinforce it by saying that the chosen group had more than the other group.

Games: You may be able to start playing simple card games, such as the game of "War." Use the dots on the cards to allow your child to compare the quantities. Another good game to practice with is "I'm Thinking of a Number," where you have a number line of cards with something hidden under one of the cards.

Number lines: Number lines are an excellent visual aid for seeing which numbers are smaller (the ones on the left) and which are larger (the ones on the right). This is a good thing to expose your child to from an early age, but be aware that both the numerals and their being in a line is fairly abstract, so it will be a while before they understand what is going on with this picture that has numbers.

2.4 COUNTING ON

Surprisingly powerful: The technique called Counting On is a simple idea that is surprisingly helpful. As your child gets comfortable with seeing a group of objects as a quantity, start doing Counting On in front of them to help them get the idea of it.

Example: Suppose your child has one row of 3 cards and another row of 4 cards. They can count all the cards one at a time "1, 2, 3, 4, 5, 6, 7" and see that there are 7 cards. However, if your child recognizes that there are three cards in one row, they don't need to count those cards. They can start their counting at 3, pointing to the three cards on the first row, and then count the second row "4, 5, 6, 7." Your child is Counting On starting at 3, and that's where the name comes from.

Understanding quantities: Being able to Count On saves time and effort. It also signals that your child is getting comfortable seeing the quantity associated with a group of objects and can work with them as a group. This is a big conceptual leap that will be very useful to them in all the adding, subtracting, multiplying, and dividing soon to come.

Connection with addition: Beginning addition is strongly linked to counting. When given a card with three dots on it, and another card with four dots on it, a child asked to add these two quantities will count them from 1 to 7. It is faster and more powerful to do this adding of 3 and 4 by starting at 3, or 4, and counting on using the other number. You will see finger techniques for doing this in upcoming lessons.

2.5 BASIC SHAPES II

Deepening explorations: Bit by bit, your child's developing skills with shapes, counting, and descriptions will allow you to include more detailed and advanced aspects of shapes. Use lots of hands-on experiences for your child. Have your child fit pieces together, pile things on top of each other, put things inside other things, and study how shapes are put together in all the objects around them. As your child experiences these things, name and discuss the shapes, and ask lots of questions about them.

Shape words: It may seem like there are a lot of words to learn. However, if you make a habit of exposing your child to these words, your child will pick them up slowly but surely.

This is a list of flat shapes:

- Parallel lines (railroad tracks, two lines in the same direction that don't meet)
- Right angle (angle found in the corner of a piece of paper)
- Circle
- Triangle
- Rectangle (looks like a piece of paper, it has four right angles)
- Square (rectangle with equal sides)
- Parallelogram (opposite sides are parallel)
- Rhombus (has four equal sides)
- Trapezoid (one pair of parallel sides)
- Pentagon (five sides)
- Hexagon (6 sides)
- Octagon (8 sides, stop sign)

This is a list of 3-dimensional shapes:

- Sphere (Ball)
- Cylinder (Round tube)
- Box (Cube)
- Pyramid (with a triangle or square as base)

Symmetries: A lot of shapes have one side that looks like the other side upon being reflected. This is called mirror symmetry. The outside shape of a human body has mirror symmetry.

Tilings: Point out tiling patterns to your child. A lot of buildings have tiling patterns on floors, walls, or ceilings. Brick walls often have interesting patterns made by their bricks. These patterns often have mirror symmetries.

2.6 ONE MORE AND ONE LESS

More important than they look: It is tempting to think of these ideas as minor, almost trivial steps. However, the ideas of One More and One Less are early and important steps for adding and subtracting.

Counting up and down: One more and one less are strongly connected to the ideas of the next number and previous number when you are counting up and down. Counting practice with counting both directions will help make it easy for your child to know which number comes next and which number was the previous one.

Adding 1 and taking away 1: Combine these ideas with adding and subtracting by asking questions such as: "How many pebbles do you have now? If I added 1 more, how many would you have? If I took 1 away so you had 1 less, how many would you have?" These are very easy and natural questions to fit into everyday conversation, and your child can talk about them without ever considering that they are doing addition and subtraction.

If you have 3 of something and your child has four of that thing, you can discuss various possibilities. Would you have the same amount if you had one more? Would you have the same amount if your child has one less. Be playful with the idea. If your numbers are 3 and 5, you can talk about getting one more twice, or about one of you getting one more and the other getting one less.

Extend from 1 to 2: When your child is ready, extend your child's understanding to what happens when you have two more or two less. There is no hurry to get to this, so please make sure your child first thoroughly understands one more and one less.

Games: With this simple bit of arithmetic, you can start playing some games with adding and subtracting. A very simple one is to play the game of Nim with 1 and 2 using either addition or subtraction. Other games are Get Out of My House and Within 1 or 2

2.7 FINGER ADD AND SUBTRACT TO 5

Adding by counting: For a long time your child has been doing adding problems by counting. If they were asked to add two things to three things, that was done by counting all five things. As your child mastered Counting On, some of that counting was replaced by starting with one of the numbers, say 3 in this example, and then counting the two remaining things. This experience with counting has also allowed your child to visualize and master the ideas of 1 more and 2 more, which has made adding 1 and 2 much easier.

Using fingers: Children of this age will benefit greatly from having manipulatives to use when doing addition. It helps cement their understanding of the numbers in terms of quantities. Of course, the manipulative that is always available to them is their fingers. When doing our example of adding two to three, they can put up two fingers on one hand, three fingers on the other hand, and bring the two hands together. Another way to do it is to raise two fingers on one hand, raise three more fingers on that same hand, and then see a total of five fingers raised.

Add 0 sometimes: Mix in adding 0 sometimes. It is easy to do, and it is important conceptually for your child.

Subtracting up to 5: The ideas behind practicing subtraction are similar to those for addition. If your child is going to subtract three from five, have your chid raise five fingers and then lower three of them. Their familiarity with one less and two less will probably make subtracting one and two very easy.

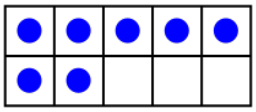
Subtract 0 and everything sometimes: Mix in questions where you subtract 0 sometimes. Also mix in questions where you subtract everything. For example, if you have three bits of food and you eat all of them, how many do you have left?

Memorizing: As you ask your child to do various adding and subtracting problems that come along, your child will become more and more familiar with them and will eventually memorize them. While it is desirable to eventually make the recall of these facts automatic and easy for your child, there is no hurry.

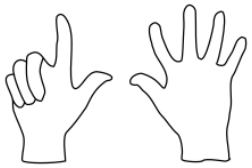
Other math facts: During this time your child's exposure to adding won't be restricted to those whose sum is 5 or less, and that is fine. For example, your child will probably have learned to add 1 or 2 to all the numbers up to 10. Your child may also have begun to learn the adding twin facts, such as 3 + 3 or 4 + 4.

2.8 FINGER ADD AND SUBTRACT TO 10

Fingers and quantities: Using your child's fingers is an effective way to do addition and subtraction to 10, and it will build your child's confidence and understanding of these operations in terms of quantities. The focus of this step is using fingers in a simple way to do adding and subtracting problems whose largest number is ten or less.



Ten frames: A ten frame is a 2 by 5 grid of open squares. These are usually filled left to right, with the top row filled first. They are useful for getting familiar with quantities up to 10 as well as seeing pairs of numbers that add up to 10 – the filled squares plus the unfilled ones will always add up to 10.



Flash numbers: A fun activity with your child is to flash a partially filled ten frame or a group of fingers on your two hands, and then have your child quickly recognize the quantity of things. If you use five or more fingers, put up five of those fingers on one hand so the two hands look like a ten frame. This activity also lays the groundwork for more easily using quantities on two hands for adding and subtracting.

Finger adding to 10: If both numbers are five or less, put up the appropriate number of fingers on the two hands and count them up. Alternatively, put up the number of fingers for one of the numbers, and then count that many more fingers for the other number. When you are done, the total number of fingers shown will be the sum.

Finger subtracting to 10: Do subtracting from numbers that are ten or less by putting up the number of fingers for the number being subtracted from. Then put down one finger at a time as you count up the number that is being subtracted. The resulting number of fingers left up is the answer.

2.9 SKIP COUNTING BY 2'S

It's fast and fun: Show your child that this is a fun way of counting that goes a lot more quickly than counting by 1's. If you pair things up and then skip count them by 2's, you will reinforce how paired things come in twos.

Getting started with skip counting: Introduce skip counting in two ways. One way is to take turns with your child as you count up to some number, alternating numbers as you go – one of you will be skip counting starting at 0, and the other will be starting at 1. After some practice, one of you stays silent and the other one counts as before, saying every other number.

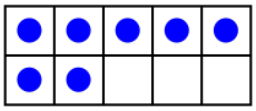
The other way for your child to get used to skip counting is to count up to some number normally the first time. Then do that counting again by saying every other number softly. Continue this practice until the softly-spoken numbers are not spoken at all.

Start at numbers other than 0: It is tempting to always start at 0. However, if you start at other numbers, such as 1, you will be helping your child with adding skills as well as multiplying and dividing skills.

Skip count downwards sometimes: Skip counting downwards helps with subtracting skills and with dividing skills.

2.10 COUNTING GROUPS

Variety: Skip counting is one way to add interest to counting a group of objects, and there are often many other ways as well. Show your child that they can use their imagination to explore the many other interesting ways they can count a picture of some objects.



Ten frames: Ten frames offer a simple example of different ways to count. Suppose you have seven dots in the usual way in a ten frame. You can count this as 5 plus 2 more. You can count by 2's from left to right to get to 4, and then add the 3 single dots. Or, you can see the 3 blank squares and count this as 3 less than 10.

Be playful: Play with all the different ways you and your child can do the counting. Seeing different ways to get the answer will open your child's eyes that math is about exploring and play, and not about arriving at an answer in some preferred way. All these different ways of counting will also develop a deeper understanding of quantities for your child.

Additive approaches: One way to count up a group of objects is to add up the different parts of the group. That's what we did in this ten frame example when we added up 5 from the top row to the 2 in the bottom row.

Skip counting: You may see a repeated pattern when you add up the parts of a group. That's what happened when we started by skip counting the two 2's on the left side.

Subtracting: Another common technique is to subtract off what is missing from what would otherwise be a whole collection. That's what we did when we subtracted 3 from 10 in this example.

Stage 3: I Can Count to 10! – Ages 3 to 6

Your child has become comfortable with counting to ten, and those numbers and quantities are now more meaningful. Your child is much better now with adding and subtracting small numbers.

This Stage extends counting to 20, and it extends the range of numbers your child can add and subtract. It also introduces the beginning of multiplying and dividing. Finally, your child's improving analytical skills and maturity warrant a discussion of early ideas for playing strategy games.

1. Mental adding by Counting On
   Adding by Counting On from one of the numbers is an important step in learning addition.
2. Mental subtracting – take away
   Subtracting by taking away and counting down is an important step in learning subtraction.
3. Mental subtracting – difference
   Another way to subtract is to find the difference between two numbers by Counting On.
4. Number bonds
   Number bonds reinforce part and whole concepts, as well as fact families for addition and subtraction.
5. Place value
   Introduce your child to the role of 10 in the numbers up to 20.
6. Fact families
   The facts 2 + 3 = 5, 3 + 2 = 5, 5 - 2 = 3, and 5 - 3 = 2 are connected. Knowing this deepens the understanding of each fact.
7. Adding twins and near twins
   Most children learn adding twins easily. This provides a foundation for doubling.
8. Multiply and Divide by 2
   This is the start of many concepts – doubling, multiplying by 2, halving, dividing by 2, and equal sharing.
9. Skip counting by 2s II
   Further develop skip counting by 2s by going up and down and starting with any number.
10. Strategy games I
    These games naturally motivate children to do problem solving.

3.1 MENTAL ADDING BY COUNTING ON

Gradual change: As your child develops skill with small adding and subtracting problems, gradually shift over to doing more of these calculations mentally. Don't be in a hurry to memorize. This time spent calculating answers by seeing the relationships between numbers will pay off handsomely in terms of a much better feel and understanding of numbers. If your child masters these facts by grinding through repeated drills with flash cards, you will have thrown away this important opportunity.

Counting on: Use 6 + 3 as an example of adding by Counting On.

Initially, your child will want to do this using their fingers, and that is fine. Say "6" and have a closed fist to indicate 0. Next, count up "7, 8, 9" and raise one finger with each number. Stop at 9, when three fingers have been raised.

Over time, using fingers with small addition problems will become less necessary. Your child will start to mentally see the change of 3 in going from 6 to 9, and will no longer need to use fingers – just as they no longer need fingers for adding 1 or 2.

Start from the larger number: With experience, your child will start to realize that it is easier and faster to start with the larger of the two numbers. For example, it is easier to do 3 + 6 as 6 Counted On with 3 more than it is do it as 3 Counted On with 6 more. Part of making this choice is realizing that the results of 3 + 6 and 6 + 3 are the same. This important observation will mean learning only about half as many addition facts!

3.2 MENTAL SUBTRACTING – TAKE AWAY

Two kinds of subtraction: There are two models for subtraction, and both are important. The first is taking away. If you are asked to subtract 3 things from 9 things, you will probably think of this as taking away or removing 3 of the things. The second model involves difference. The difference between two numbers is the distance between them . If you are asked to subtract 7 from 9, you are likely to do this be finding how far apart they are, what their difference is.

Both understandings of subtraction are needed and are useful. For mental arithmetic, having a choice of using take away or difference can make a big difference in the ease of the calculation. Additionally, some problems are more naturally understood as take away or difference problems.

Counting down: Counting down is a natural way to do take away. Use 9 - 3 as an example of subtracting by counting down.

As with adding, your child may want to initially do this using their fingers, and that is fine. Say "9" and have a closed fist to indicate 0. Next, count down "8, 7, 6" and raise one finger with each number. Stop at 6, when three fingers have been raised.

Over time, using fingers with small subtraction problems will become less necessary. Your child will start to mentally see the change of 3 in going from 9 to 6, and will no longer need to use fingers – just as they no longer need fingers for subtracting 1 or 2.

Subtracting 1 to 4: With practice, your child will build up speed using this approach for subtracting the numbers from 1 to 4. Take your time and make sure everyone enjoys the process.

3.3 MENTAL SUBTRACTING – DIFFERENCE

Distance or gap size: If you are asked to find the difference between two numbers, you are doing a subtraction problem. Finding the difference is understood as finding the distance between the numbers, and this can be calculated by asking what number needs to be added to span the gap between them.

Counting On: Use finding the difference of 9 and 6 as an example. Calculate this by keeping track of how many numbers you use as you Count On from 6 to 9. As with the instructions for adding in a previous learning step, this can be done with or without fingers. If your child has been practicing adding problems, they can probably very quickly count "7, 8, 9" and see that the gap is 3.

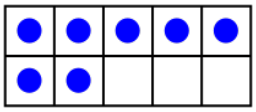
Subtracting and adding: Using Counting On to find the difference is a great way to see the connection between adding and subtracting. When we Counted On starting at 6 to add 6 + 3, we wanted to find the result of Counting On three times. When we Counted On from 6 to find 9 - 6, we knew the result of 9 and wanted to find out what we needed to add to 6 to get there.

Differences of 1 to 4: With practice, your child will build up speed using this approach for finding differences of sizes 1 to 4. Take your time and make sure everyone enjoys the process.

3.4 NUMBER BONDS

Part and whole: Seeing a whole thing as being made up of its parts is an important developmental step for a child. The number bonds for a number, say 6, are all the ways of pairing two numbers that add up to 6. It is all the ways of taking the whole of 6 and breaking it into two parts. This reinforces an understanding of the connection between addition and subtraction as forming groups of fact families – a topic to be covered in an upcoming learning step.

What's missing?: Let's continue using number bonds for 6 as our example. The number bonds for 6 are: 0 + 6, 1 + 5, 2 + 4, and 3 + 3. A child who has learned these well will then have little trouble answering the question: "What do I need to add to 2 to get 6?" They will know that 2 + 4 is a number bond for 6, so 4 is the part of 6 that is missing.



Ten frames: Fluency with number bonds for all numbers up to about 12 is very useful for doing adding and subtracting, The number bonds that arise the most often are those for 10. Ten frames are designed to help visualize number bonds for 10. A ten frame with 7 dots in it makes it visually obvious that 7 + 3 is a number bond for 10.

Games and puzzles: There are quite a few games and puzzles that involve number bonds. Sum Groups is a puzzle designed to practice number bonds. Many games, such as Memory Challenge and Go Fish, have versions that use target totals and thereby exercise number bonds.

3.5 PLACE VALUE

Meaning for the numbers 10 to 20: At first, the numbers above 9 can be thought of as just the next numbers that come along. There is nothing wrong with that, and that viewpoint will work just fine for a long time. Eventually, with numbers all the way up to 100 coming on the horizon, it will be time to Introduce your child to the role of 10 in the numbers from 10 to 20.

Adding and subtracting 10: Start this introduction by helping your child learn how to add 10 to single-digit numbers and subtract 10 from double-digit numbers from 10 to 20.

Practice with physical quantities: Find something you have a lot of – perhaps pebbles, sticks, or toothpicks. Let's use 3 and 13 as an example. If you place 13 things on a flat surface, you can separate them into one group of 10 and one group of 3. This demonstrates both that 3 + 10 is 13, and also that 13 - 10 is 3.

Numerals: If your child is ready to use numerals, write down 13 = 10 + 3 and 13 - 10 = 3 as you do the demonstration just described. When we write a two-digit number, such as 13, the place on the left is the tens place and the place on the right is the ones place. Then tens place tells how many tens the number has, and the ones place how many ones it has. For 13, it is made up of 1 ten and 3 ones, so 13 = 10 + 3. As another example, 20 is made up of 2 tens and 0 ones.

It takes time: Breaking up a number into tens and ones, particularly symbolically with written numbers, is a big conceptual step and you should expect this to take many demonstrations, many explanations, and lots of time. As with so many of these foundational elements, your child will absorb it eventually and there is no hurry.

3.6 FACT FAMILIES

A family of facts: Addition and subtraction facts are easily grouped into facts that are closely related. Take this family of facts for example: 2 +3 = 5, 3 + 2 = 5, 5 - 3 = 2, and 5 - 2 = 3. The picture below shows how these are interconnected.

Use this kind of picture with your child. It shows one group of 3 dots, another group of 2 dots, and all together there are 5 dots. You can add the groups of dots in either order – either as 3 + 2 or 2 + 3 – and get the same result. If you take the 5 dots and you cover over one group, say the group of 2, then it is clear why 5 - 2 = 3.

Adding and subtracting are related: Fact families and their associated pictures make it very clear how closely tied together addition and subtraction are. One important part of their relationship is that they undo each other. If you start with 3 and add 2, and then subtract 2, you are back with having the original 3. Similarly, if you start with 5 and subtract 2, and then add 2, you are back to having the original 5.

Addition order doesn't matter: Another important thing to reinforce about fact families is that they show that the order that addition is done in does not change the result. So, if your child is asked to do 4 + 8 and they prefer to do 8 + 4, then they can do that.

3.7 ADDING TWINS AND NEAR TWINS

Children like them: Adding twins are when a number is added to itself, such as 3 + 3. Children generally enjoy adding twins, so they learn them eagerly and readily. The sums are also the numbers you get when you skip count by 2's, so that helps to reinforce these adding facts.

A foundation for doubling: Knowing adding twins quickly leads to knowing how to double. Suppose you ask your child to double 3. They know that doubling something means to take two if it. So, doubling 3 means 3 plus 3, which is 6.

Near twins: A pair of numbers is a near twin if they are one apart, such as 3 + 4. Once your child knows their adding twins, near twins is an easy next step. Think of 3 + 4 as 3 + 3 + 1, so it is 1 more than 3 + 3. So, 3 + 4 = 6 + 1 = 7. Alternatively, think of 3 + 4 as (4 - 1) + 3, which is 1 less than 4 + 4. So, 3 + 4 = 8 - 1 = 7. Allow your child to pick whichever approach they prefer for each near twin, or have fun doing them both ways.

This technique of using a familiar math fact to figure out a closely related math fact is powerful. It will help with learning many math facts in the months to come. Take your time and make sure your child thoroughly understands how this works. To explain it, demonstrate how 3 objects and 4 objects can be split up as 3 objects plus 3 objects plus one more – being able to see and touch things at this age is very powerful.

3.8 MULTIPLY AND DIVIDE BY 2

Connected concepts: In the previous learning step we discussed doubling. This is connected to a surprising number of related concepts, as you will soon see.

Multiplying by 2: Multiplying by 2 is the same as doubling. However, it uses interesting new wording that your child will need to get used to. When you first introduce the phrase of multiplying by 2, be sure to mix in the use of doubling from time to time to help with the transition. Your child will be getting used to the new words "multiplying" and "times."

Half of something: When something is doubled, such as 3 doubled is 6, then you can take half of the result and get back the original number. For example, suppose you picture 3 doubled as two rows of 3 things. Then taking half of those two matched rows is to take one of the rows, which is 3 things.

Equal sharing: When you talk about taking half of something, you can combine that with the idea of sharing it equally among two people. If two people are to get equal shares, they will each get the same thing and that will be half of the original amount.

Dividing by 2: If your child is comfortable with all the concepts so far in this learning step, then they are ready to talk about dividing! You can start talking about dividing something equally, or dividing it in two, or dividing by two. Mix together using all these phrases with your child – it will take time for them to absorb this new vocabulary. As far as the math is concerned, they have mastered all the ideas already!

Concrete examples: Do lots of practice with each of these new ideas and words with groups of objects. Make your child's understanding be concrete rather than simply abstract. Anytime there is something to share among two people is a perfect time to practice. You can of course start extending this to sharing among more than two people if you like.

The beginning of multiplying and dividing: As you have seen, this is the starting point of multiplying and dividing. There are one or two new ideas, and a lot of new vocabulary. It's exciting to see your child step into this new part of the world!

3.9 SKIP COUNTING BY 2'S II

New connections: Your child started skip counting by 2's during Stage 2. Their skip counting is now going higher and faster, and it is also now connected with new ideas, such as with multiplying and with even and odd numbers.

Multiplying by 2: For example, multiplying something by 2 means to take that many 2's. Suppose your child is calculating 5 times 2. This means adding 2 to itself 5 times, which is exactly what your child will get if they skip count by 2's 5 times starting at 0. Of course, they can also calculate this quantity by doubling the number, in this case 5.

Start anywhere and go up or down: It is important to keep practicing skip counting by 2's starting anywhere and going up and down. This practice will be surprisingly helpful for adding, subtracting, multiplying, and dividing. Get used to doing this practice anytime you have a somewhat large collection of things to count.

Even and odd numbers: An even number is one that can be split in half evenly. It is a quantity that two people can share and have nothing left over. The odd numbers are the other numbers, the numbers that cannot be shared evenly. Even numbers are the numbers you get when you skip count by 2's starting at 0. Odd numbers are the numbers you get when you skip count by 2's starting at 1.

Skip count by other numbers: If you like, this is a good time to start skip counting by other numbers. Skip counting by 10's, 11's, and 5's are good starting places.

3.10 STRATEGY GAMES

Developmental stage: Strategy games are games where the players have choices that produce better or worse outcomes. These games have a lot to offer children. In particular, they naturally motivate children to do problem solving. However, the difficulty with these is that they require rule following and a willingness to participate in organized play. If your child is not ready yet for this kind of play environment, please be patient and let them get to a developmental point where they are ready. There is plenty of time.

Be patient: When children are first ready to play organized games and are introduced to a new game, you will probably need to be very patient. It could well be that the first ten times you attempt to play the game your child will have fun with all the pieces and do many things that have nothing to do with the game. This is a process. Your child needs to become familiar with the game and its pieces, and the novelty needs to wear off a bit. You will know when your child is ready and your patience has paid off.

Play around: At first, it is likely that your child won't really have a sense of winning and losing, and won't understand that their choices make a difference. That's okay. You are just there to have fun together. Over time, as the game and its choices become more familiar, your child will start to see that their choices make a difference and that they care about those differences.

Invisible math sometimes: Strategy games don't always involve numeric components, but they are mathematical anyway. Any strategy game offers many important learning opportunities, some mathematical and some not. As mentioned earlier, problem solving is high on the list of skills that are developed by these games. Critical thinking and logical communication are also being developed. The rule following and turn taking are very good for social-emotional learning.

Stage 4: I Can Count to 20! – Ages 4 to 7

During this Stage, your child will extend their counting to 100. As their counting develops, they need to get a solid understanding of double-digit place value. They will also come to master all their single-digit addition and subtraction facts. This is also an exciting time when multiplication by the numbers from 1 to 5 starts to be mastered.

1. Counting to 100
   Count both forward and backward to solidify an understanding of these numbers.
2. Two-digit place value
   Use expanded form to understand place value and how to compare numbers.
3. Finger Addition
   Learn finger addition doing it the Easy Way.
4. Finger Subtraction
   Learn finger subtraction doing it the Easy Way.
5. Compensation for Add and Subtract
   Compensation is a useful tool for simplifying addition and subtraction calculations.
6. 10 as a midway point
   Use 10 as a midway point for addition and subtraction.
7. Skip counting by 2s, 5s, 10s
   Practice this skip counting going up and down, starting anywhere.
8. Beginning multiplying
   Gently introduce the words "multiplying" and "times" to describe doubling and skip counting.
9. Multiplying 1-5 by 1-5
   At this point, your child knows all the multiplication facts for 1 to 5.
10. Strategy games II
    Discuss deeper ideas about strategy games.

4.1 COUNTING TO 100

10's: When your child's counting extends from 20 up to 100, the hard parts are the changes in 10's. Those are the parts to give special support and practice. Once your child knows the sequence of the 10's, it is usually very easy for them to fill in the changes in 1's. For example, they will quickly learn to count "70, 71, 72, 73, 74, 75, 76, 77, 78, 79."

Skip count by 10's: When your child gets stuck counting between 1 and 100, it is usually at a transition in 10's. Typical questions to help your child get unstuck are: "What comes next after the 60's?" or "What comes before the 50's?" Practice skip counting by 10's to help with this. If your child can produce the 10's from 0 to 100 forward and backward, then they will be ready to answer those questions about which 10's come next or before.

Use objects: Make this counting have more meaning by practicing it using a large group of small objects, such as pebbles. Gather together 100 of these objects and put them in a big pile off to the side. As your child counts up from one, pull over one object at a time with each number and include it in the current group of objects. Each time a group of ten objects is formed, put that group together in a special tens area. As the numbers increase, the tens area will have more and more groups of tens. By the time your child is in the 50's, there should be five groups of ten and a small collection of left over ones.

You can also do this practice in reverse by starting with ten groups of tens and then taking away one at a time as your child counts down from 100.

Count both directions: Children often become really good counting quickly and automatically upward from 1 to 100, but then have trouble counting in the opposite direction. Counting from 100 to 1 forces more thought about the 10's transitions and helps with understanding the numbers better in general.

100 number chart: Having a ten by ten 100 number chart going from 1 to 100, or from 0 to 99, will help your child see the pattern of the numbers. It will make it particularly clear how the tens place stays the same for ten numbers and the ones place keeps changing.

4.2 TWO-DIGIT PLACE VALUE

Number meaning: Seeing a two-digit number as some number of tens and ones is central to understanding and working with those numbers. Doing two-digit addition and subtraction requires fluency with the concepts of place value. Develop that understanding through lots of experiences with physical objects and written numbers.

Bundles of tens: Take something you have lots of, make a large pile of it, and put some of them in a few bundles of ten. For example, out of that large pile ask your child to gather 23 of them. Then ask your child to group them in groups of ten. They will form two groups of ten and 3 singles. See what happens with the groups of ten in these 23 things as you put in some additional items or take some away, always making sure that there are no more than 9 single items at any time.

Expanded form: The expanded form of a number is writing it as a sum of its place value parts. For example, 23 would be written as 20 + 3, and 256 would be written as 200 + 50 + 6. Practice with your child converting in both directions between a number's usual form and its expanded form. Point out that the expanded form of a number is the same as taking that many things and bundling them into groups of ten and have ones left over.

Counting with tick marks: Counting using groups of five tick marks is a natural thing people like to do and is strongly related to place value. If you are counting 23 things with tick marks, you will end up with 4 bundles of 5 and 3 tick marks left over. Those 4 bundles of 5 can be reorganized as 2 bundles of 2 groups of 5, which is our expanded form view of 23.

Skip counting: Skip counting by various numbers is a way to build mental practice with adding or subtracting single-digit and double-digit numbers. Although it is a great way to practice, this does require some mental steps that not every child will be ready for at this point – there is no hurry. Here are two examples. With lots of practice, these thinking steps will become automatic.

Skip count upward by 8 starting at 23. Think of 23 as 20 plus 3 more. Using the number bonds for 10, 3 will need 7 more to make a group of 10. Use 7 out of the 8 being added so that 3 plus 7 forms another group of 10. So, 23 + 8 becomes 20 + 10 + 1, which is 31.

Skip count downward by 5 starting at 23. Break up 5 into 3 and 2. To subtract 5, we will first take away 3 and then take away 2. 23 subtract 3 brings us to 20, which we will think of as 10 + 10. Subtracting the remaining 2 from one of those 10's leaves 8, so our answer is 10 + 8, which is 18.

4.3 FINGER ADDITION THE EASY WAY

Counting On using fingers: Any single-digit number can be added to any number using this method. This uses the child's hands to keep track of the single-digit number that is being added. It makes use of Counting On to find the final total, and also your child's skill with recognizing quantities of fingers.

Example: Use 8 + 7 as an example. We could use either number as the number we start with, but it will be quicker and less work if we start with the larger number, which is 8 in this case. Start with no fingers raised and both hands in a closed fist. Your child is going to count starting at 8, and each time a new number is mentioned, your child will raise another finger. So, starting with "8" with no fingers raised, the child Counts On and says "9, 10, 11, 12, 13, 14, 15" while raising one more finger each time. Your child will stop at 15 because they will recognize they have 7 fingers raised at that point.

Any starting number: Notice that your child can use this method with any starting number. This example would have worked just as easily for adding 58 + 7.

It is reliable, but it will be replaced: This will eventually be replaced by other methods, but for now this is reliable and is something your child can be assured will always give them the right answer any time they need it.

4.4 FINGER SUBTRACTION THE EASY WAY

Two subtractions: The two models for subtraction, take away and difference, are both important and both need to be practiced. Here are methods that use a child's hands to keep track of part of the calculation. As with finger addition using fingers, these methods use your child's skill with recognizing quantities of fingers. We'll use 14 - 8 as our example for both methods.

Take away with fingers: This method uses counting down to take away a single-digit number from any number. Have your child start with two closed fists and say "14." Counting down from 14, your child raises one more finger with each new number: "13, 12, 11, 10, 9, 8, 7, 6." Your child will stop at 6 when they see they have 8 fingers raised.

Difference with fingers: This method uses Counting On to find a single-digit difference between any two numbers. Have your child start with two closed fists and say "8." Counting On from 8, your child raises one more finger with each new number: "9, 10, 11, 12, 13, 14." When your child hits 14, they look at their fingers and see that the difference is 6.

Reliable but will be replaced: These will eventually be replaced by other methods, but for now these are reliable and are something your child can be assured will always give them the right answer any time they need it.

4.5 COMPENSATION FOR ADD & SUBTRACT

Useful and easier than you think: Compensation is a useful mental arithmetic tool for simplifying addition and subtraction calculations of all sizes. Understanding it also increases number sense for addition and subtraction. It is simpler than it sounds.

For example, suppose you were adding 99 + 15. You would recognize that 99 needs just one more to make 100, a number that is much easier to work with than 99. So you would move 1 from the 15 to the 99 – you would have the same total number of things, but they are distributed in a way that's easier to work with. This problem becomes 100 + 14, which is very easy to do. This is the kind of thing we'll be doing.

Compensation for addition: The idea is to give or take some small amount to make one of the numbers easier to work with. We will typically be making one of the numbers into a multiple of 10. Suppose you are adding 8 + 7. The 8 just needs 2 more to become 10, so take that 2 away from the 7. This makes 8 + 7 into 10 + 5, which is easy. We could also have done this problem by giving 3 to the 7 to make it 10. In that case, we'd turn 8 + 7 into 5 + 10.

More compensation for addition: There are other possibilities for using compensation in addition problems. Consider 6 + 8 for example. The 6 could give 2 to the 8 to make this problem 4 + 10. However, the 8 could give 1 to the 6 to make this 7 + 7, an adding twins problem. Challenge each other to think of different ways for doing a given adding problem.

Compensation for subtraction: For subtraction, we will add the same amount or subtract the same amount from both numbers. This will keep the distance between them the same but will make them easier to work with. Typically that will mean turning the number we're subtracting into a multiple of 10. Suppose we are subtracting 13 - 8. If we add 2 to both numbers, then the distance between stays the same, but now we are subtracting 15 - 10, which is easy. Similarly, if we were asked to do 17 - 13, we could subtract 3 from both numbers and turn it into 14 - 10. Alternatively, we could subtract 10 from both numbers and turn it into 7 - 3.

4.6 10 AS A MIDWAY POINT

Number bonds for 10: The number 10 is often a handy mental arithmetic midway point to use for doing addition and subtraction problems that involve numbers above 10. Your child should have a strong command of the number bonds for 10 to take full advantage of these methods.

Addition sums over 10: Suppose your child is given the addition problem 5 + 7. The number bond for 10 using 7 is 3, so your child can use 3 of the 5 to get to 10. The remaining 2 of the 5 will bring the total to 12. The point was to break up the 5 into two pieces, 3 and 2 – one which got the 7 to 10, and the other gets added onto the 10. This problem could also have been done the other way. The 7 could have been broken up into 5 and 2 – the 5 would be added to the original 5 to get 10, and then the 2 would be added to 10 to get 12.

Notice that this is similar to the thinking for doing addition compensation.

Subtracting from a number bigger than 10: Let's use 12 - 7 as our example. We can do this as a take away or a difference problem.

As a take away problem, we'll use 2 of the 7 to get 12 down to 10. We then have 5 of the 7 left to take the 10 down to 5. We broke 7 into 2 and 5 to be able to use 10 as an intermediate stop along the way.

As a difference problem, the total distance between 12 and 7 is the distance between 12 and 10 plus the distance between 10 and 7. The distance between 12 and 10 is 2, and the distance from 10 and 7 is 3, so the total distance is 2 plus 3, which is 5.

4.7 SKIP COUNTING BY 2'S, 5'S, AND 10'S

The easy ones: At this point your child should be very comfortable skip counting by 2's, starting from anywhere and going up or down. If your child has not started already, it is time to extend this to skip counting by other numbers. The two easiest ones are skip counting by 5's and 10's.

By 10's: Not only is skip counting by 10's easy to do, it also provides practice with place value. Your child will quickly realize that, if they skip count by 10's starting at 3, all the numbers will have a 3 in the ones place and the only thing that changes is the tens place. If you have a 100 chart available, use it to show your child that all the numbers go down or up in one column of the chart.

By 5's: After your child masters skip counting by 10's, it is time to do the 5's. When your child skip counts by 5's, they will be comforted to see that every other number is 10 apart, as if they were skip counting by 10's.

Experiment with others: There is no hurry to do the other numbers. Skip counting by 9's can be fun because the ones place gets one smaller and the tens place gets one bigger with each step. Skip counting by 11's is quite easy.

Have fun with it: This is something you can make into an activity to do with two or more people. Someone says a number to start with, what the skip size will be, and whether to go up or down. Then go around the group with each person saying the next number.

4.8 BEGINNING MULTIPLYING

New words: During the last half of Stage 4, the learning steps involving doubling described starting to use the words "multiply" and "times." If you haven't done so already, start using these words more and more. You now have plenty of situations to use those words in.

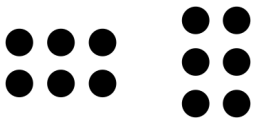
Doubling and tripling: Doubling is multiplying by 2, and tripling is multiplying by 3, and you should start calling it that. If you are to double some number, say 4, ask your child to multiply it by 2 or ask them what 2 times 4 is. They have all the tools they need, they just need to adjust to the new wording.

Skip counting is multiplying: If your child skip counts by 5's seven times starting at 0, your child will have seven 5's. That is the same as multiplying 7 times 5. From now on, anytime your child is asked to do a multiplication problem, they can skip count to find the answer. Over time, they will have better ways to find the answer, but this is a secure method they can use for now.

4.9 MULTIPLYING 1-5 BY 1-5

It's exciting: Your child now has all the tools they need to master multiplying any number from 1 to 5 by any number from 1 to 5. That is an exciting place to be for a young child. Let's review all the things they know that support this.

Order does not matter: When you multiply, it doesn't matter whether you multiply 3 time 4 or 4 times 3, the result is the same. This is a big work saver and it allows your child to pick their favorite way to multiply things. For example, if they want to calculate 2 times 5 by doubling 5, that's fine. However, if they prefer to skip count by 2's five times, that's fine as well.



Have your child look at a picture of 2 rows of 3 dots. Whether you have 2 rows of 3 dots, or 3 rows of 2 dots, you have the same total of 6 dots. All you need to do is turn the picture half way around to see that they are the same picture.

Multiplying by 2 is doubling: Multiplying by 2 is the same as doubling, so your child has that skill already.

Multiplying by 3 is doubling plus one more: If you have 3 of something, that's the same as having 2 of them and then adding one more. If your child needs to multiply 3 times 4, they can double 4 to get 8, and then add one more 4 to that to get 12. Alternatively, you can also skip count by 3's four times, or skip count by 4's three times, if you wish.

Multiplying by 4 is doubling twice: If you want to get 4 of something, start by doubling it to get 2 of it, and then double that to get 4 of it. For example, to multiply 4 times 5, double 5 to get 10, and then double 10 to get 20.

Multiplying by 5 by skip counting: Skip counting by 5's is so much fun, that is probably the preferred method. Also, all the 5's end in either 0 or 5, so that makes them pretty easy to remember.

4.10 STRATEGY GAMES II

The value of these games: Strategy games are games where the players have choices that produce better or worse outcomes. These games have a lot to offer children mathematically, even when there is no explicit numerical content. In particular, they naturally motivate children to do problem solving.

An introduction to Nim: Now that your child has been playing strategy games for a while, it is time to look into how to get more out of them. Let's use the game of Nim as an example. This game has some very simple rules: Pick a starting number, say 10, and a person to go first. Players alternate turns choosing to subtract 1 or 2 from the running total. The person who reaches 0 wins. The arithmetic is simple enough, but the strategy is challenging.

Learning from experience: If you imagine you are playing any game against an extremely capable opponent, the game becomes a puzzle. How can I find the move that let's me have my best chance of winning no matter how well my opponent plays? One strategy is to play the game many times and pay attention to what seems to work and what doesn't. This approach is a good start and it does provide the opportunity for observation and insights. This can be a slow method of learning, and it may be very hard to find patterns in complicated games – imagine all the possibilities if we picked the starting number of 100 for Nim!

Solving Nim: Can we look ahead a few moves and figure out good moves that way? That is hard to do starting at 10. If we start at 5, it's pretty easy to do. If we subtract 2, the new number will be 3. No matter whether the other play subtracts 1 or 2 at this point, we will win. So, if we are at the number 5, we know how to win. What happens with other small starting numbers? Can we find a pattern that will tell us which starting numbers will be winners and which will be losers? Can we explain why that pattern is true?

The goal: If we follow this line of attack, we will completely solve how to play the game of Nim. What worked for Nim may or may not work for another game. This is problem solving, and each new problem can bring fresh challenges and a need for new ideas. And that is the fun. Share this attitude of puzzling and being challenged to your child. As they play a strategy game, discuss with them their ideas about what would make for a better or worse move at that moment. The point isn't so much to find the perfect move as it is to enjoy looking for it and having a discussion about it with people who they can enjoy sharing ideas with.

Stage 5: I Can Count to 100! – Ages 5 to 8

In this Stage counting progresses into the three-digit numbers beyond 100. The ideas of place value become incresingly important, and the use of expanded form for numbers can make many of these ideas clearer. Now that single-digit addition and subtraction have been mastered, it is time to learn single-digit multiplication and dividision.

1. 3-digit place value
   Extend the use of expanded form to 3-digit numbers to understand place value.
2. Double-digit add/sub
   Use expanded form to see how 2-digit addition and subtraction work.
3. Skip counting by 2'S to 10'S
   Practice skip counting up and down starting anywhere and by any number from 2 to 10.
4. Multiplication – 2, 4, 8, 5, 10
   These involve doubling and multiples of 5. They are quick to learn and provide a good frame-work for the remaining numbers.
5. Multiplication – 3, 4, 6, 9, 11
   Use the idea of one more or one less to learn these based on the earlier numbers.
6. Single-digit multiplication
   Your child will now know all of single-digit multiplication!
7. Divisors, factors, and multiples
   Introduce the terms divisors, factors, and multiples.
8. Primes, composites, and powers
   Learn about primes, composite numbers, and units. Practice prime factorizations – these often involve repeated factors, which is a good time to learn about powers of numbers.
9. Fact families II
   Group multiplication and division facts by families.
10. Single-digit division
    Your work with skip counting, learning multiplication facts, and fact families will smooth the way for dividing both with and without remainders.

5.1 3-DIGIT PLACE VALUE

Many objects: Relating math concepts to physical objects is often the best approach for young children. The difficulty here is that it is not so easy to have hundreds of objects in the first place, and it can also be unwieldy to deal with such a large collection. In addition to having groups of ones and tens of some object, one strategy is to use symbolic place holders for large groups, such as 100s. You could have several pieces of paper or wood on which you have written "100."

Use objects: Practice having your child represent various quantities using objects grouped in ones, tens, and hundreds. For example, ask how 325 would be represented using these objects. Include examples such as 206, 430, and 500 that have zero things in one or more categories. Also, lay out collections of ones, tens, and hundreds and ask your child to name the number for this quantity.

Use expanded form: Once your child is thoroughly comfortable with the connection between numbers and their quantities, start using numerals and expanded form to represent the numbers. Take a number, such as 325, for example. Represent it with quantities for hundreds, tens, and ones, and then use the quantity representation to write the number in expanded form as 325 = 300 + 20 + 5. Do this same practice in the opposite direction by writing 100 + 40 + 6, representing it with physical groups of hundreds, tens, and ones of objects, and then asking what the total number is.

5.2 DOUBLE-DIGIT ADD/SUBTRACT

Single digit: A gentle introduction to working with two two-digit numbers is to have one of them be a single-digit number. One of the best ways to practice adding or subtracting a single-digit number with a double-digit number is to do skip counting using various skip sizes, going up or down, and starting at any number. The following methods for two two-digit numbers will of course work when one of the numbers is a single-digit number.

Use objects: Grounding your child's understanding by using physical objects is always a good idea. Start by representing both of the numbers with groups of tens and ones.

Adding: If you are adding the two numbers, put all the groups together and discuss the result. If you are adding two numbers, such as 23 and 45, that is all you need to do. However, if you are adding 23 and 48, the two groups of ones form at least a group of ten. Talk about how that changes the total number of tens you have from 6 to 7. This is called regrouping.

Subtracting: If you are subtracting, start by removing the tens being subtracted from the bigger number and then attempt to remove the appropriate number of ones. For example, if you are subtracting 23 from 45, then there are enough ones and you are done. If there are not enough ones, such as if you are subtracting 28 from 45, then discuss how one of the groups of tens needs to be broken up and included with the ones. The original grouping of 45 as 4 tens and 5 ones becomes a grouping of 3 tens and 15 ones. This is also called regrouping.

Use expanded form: Use objects for double-digit adding and subtracting until your child thoroughly understands the process and why regrouping makes sense when it is needed. At this point, move to representing the adding and subtracting with numbers written in expanded form. When working with numbers in expanded form, the process and the steps are exactly the same as they were when you were working with groups of tens and ones – and that is the point.

Making it automatic: Over time and with lots of practice, your child will stop needing to use groups of tens and ones or expanded form. However, as with so many other things, there is no hurry getting to that point – it will come with practice.

5.3 SKIP COUNTING BY 2'S TO 10'S

Practice: Practice skip counting up and down starting anywhere and by any number from 2 to 10. Not surprisingly, this is valuable for learning multiplication and division. It is also very helpful for getting better at mental addition and subtraction. One of the nice things about skip counting is that it can be done anywhere and at any time that you have some spare time.

Patterns on a 100 chart: Look for patterns that show up as your child skip counts. This is most easily done using a 100 chart, but you can also do it by writing the numbers in a column and watching what happens to the ones and tens digits as they march down the column. Some numbers, such as 8 and 9, have interesting patterns in their ones digits, and other numbers, such as 3, are not as interesting.

Make it fun: This is something you can make into an activity to do with two or more people. Someone says a number to start with, what the skip size will be, and whether to go up or down. Then go around the group with each person saying the next skip count number.

5.4 MULTIPLYING BY 2, 4, 8, 5, AND 10

Good framework: These numbers are usually quick to learn, and once learned they provide a good framework for learning the remaining numbers.

Multiplying by 5 and 10: Knowing how to multiply by 10 is quick to learn, and it is important for understanding place value. It also can make it easier to learn how to multiply by 5.

The 5's can either be learned by skip counting by 5 until they become automatic, or they can be learned using the 10's. If you multiply 6 times 5, that will give you half as many 10's. Half of 6 is 3, so the answer is 30. If you multiply 7 times 5, you need to put one of the 5's off to the side and proceed as you would for 6 times 5. The answer for 6 times 5 is 30, and then adding in the 5 held in reserve gives the answer of 35.

Multiplying by 2, 4, and 8: These three can be done with lots of doubling. Your child should have had lots of practice with doubling and multiplying by 2. Multiply by 4 either by skip counting or by doubling the answer from multiplying by 2. For example 4 times 3 is twice as much as 2 x 3, so the answer is 6 doubled, which is 12. Multiply by 8 by skip counting by 8 or by doubling the answer from multiplying by 4.

5.5 MULTIPLYING BY 3, 4, 6, 9, AND 11

Several strategies: These are grouped here because they can be done with the idea of using one more or one less, which is described in a moment. However, several of these can be done in other ways. They can all be done with skip counting, if your child prefers that approach. Multiplying by 4 is the double of multiplying by 2. Multiplying by 6 is the double of multiplying by 3. Multiplying by 11 is quite simple and barely needs any practice.

Multiplying by 9 has a special rule, that some children enjoy. Use 6 times 9 as an example. For the answer, put one less than the number in the tens place (which is 5), and subtract the tens place from 9 to get the ones place (which is 4). So 6 times 9 is 54. As you'll see, this is no different than subtracting 6 from 6 times 10, but somehow it feels like more fun.

One more and one less: Use the idea of one more or one less to learn these based on other numbers you now know how to multiply by. The numbers 3, 6, and 11 are one more than numbers you already know. For example, 6 times 7 is one more 7 than 5 times 7. So 6 times 7 is 7 + 35, which is 42.

The numbers 4 and 9 are one less than numbers you already know. For example, 4 times 7 is one less 7 than 5 times 7. So 4 times 7 is 35 - 7, which is 28.

5.6 SINGLE-DIGIT MULTIPLICATION

Missing piecees: If your child memorizes the one or two remaining multiplication facts, they will then know all of single-digit multiplication! For example, they may not know what 7 times 7 is yet. Similar to adding twins, squares are a fun category for many children, and these can be practiced on their own. Keep all this practice light-hearted and do not get too goal oriented.

Pulling it all together and memorizing: Bit by bit, with practice and repeated exposure, your child will memorize all the multiplication facts. While it is important that these eventually become easy and automatic for your child, it is not essential that this happens quickly. It is far more important that you make this fun and that your child enjoys seeing how the various multiplication facts interrelate – which ones are doubles or halves of others, which ones are one more or one less than others, and which ones have interesting patterns with their ones digits.

Limited and fun flashcards: Used sparingly and lightheartedly, flash cards, or something similar, can be helpful. If your child has trouble remembering a handful of the multiplication facts, make a list of just those facts so they can be briefly practiced on their own.

5.7 DIVISORS, FACTORS, AND MULTIPLES

Divisors and factors: If a number evenly divides into a number, then it is called a divisor of it. For example, 3 is a divisor of 6 because 3 divides into 6 exactly 2 times. 4 is not a divisor by 6 because it goes into it 1 1/2 times. The word factor means the same thing as divisor.

Common divisors: In some mathematical situations, particularly with simplifying fractions, it is useful to find numbers that evenly divide two given numbers. Such numbers are called common divisors or common factors. The common divisors of 20 and 8 are 1, 2, and 4. You may enjoy exploring together why all the common divisors for a pair of numbers are divisors of the greatest of the common divisors.

Multiples: A multiple of a number is anything that can be produced by multiplying the number by a whole number. For example, some multiples of 6 are 0, 6, 12, and 18. Notice that any multiple of a number has that number as a divisor. For example, each of the multiples of 6 has 6 as a divisor.

Common multiples: A number which is a multiple of two given numbers is said to be a common multiple for them. Some common multiples of 6 and 4 are 0, 12, 24, and 36. Notice that all the common multiples are multiples of the least of the positive common multiples. Common multiples will be useful in adding and subtracting fractions.

Introduce these words: Slowly introduce these new words to your child as you discuss situations that involve multiplication and division. They are useful words that simplify many discussions once the words are understood.

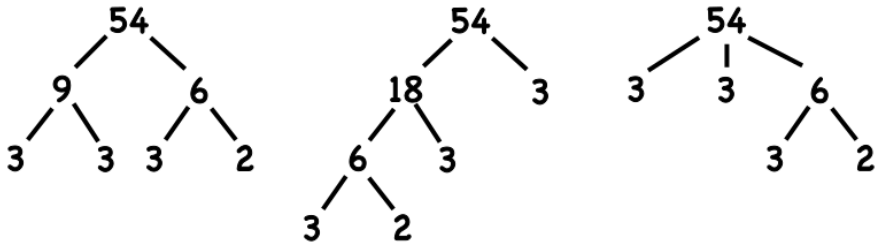
5.8 PRIMES, COMPOSITES, AND POWERS

Primes: Primes are central to understanding multiplication and division of whole numbers. As you will see, primes are the building blocks of numbers using multiplication. A prime number is a number larger than 1 whose only divisors are 1 and itself. The numbers 2, 3, 5, 7, and 11 are the first few prime numbers.

Composites and 1: There are three kinds of positive whole numbers: 1 (which is called a unit), primes, and composites. Composites can be thought of as being constructed from primes. For example, 12 is 2 times 2 times 3. Every number larger than 1 is either a prime or can be uniquely written as a product of two or more primes.

Prime factorizations: Getting to know prime factorizations really well will be very helpful for many parts of the math your child is about to learn. Repeating the prime factorizations of the numbers up to 20, or even 30, is a good exercize for getting to know these factorizations. Simply go through the list of numbers in order like this: 1 - unit, 2 - prime, 3 - prime, 4 - 2 times 2, 5 - prime, 6 - 2 times 3, 7 - prime, 8 - 2 times 2 times 2, 9 - 3 times 3, and 10 - 2 times 5.

Powers: Prime factorizations often involve repeated prime factors, so this is a good time to learn about powers and to practice them. It is quicker and easier to understand to say "2 to the fourth" than it is to say "2 times 2 times 2 times 2." 2 squared means 2 times 2, and 2 cubed means 2 times 2 times 2.



Factors and factor trees: For larger numbers, it may not be immediately obvious what the prime factorization is. For these numbers, find one of the factors and use that to break apart the problem into easier pieces. For example, 54 is 9 times 6. Because 9 is 3 squared and 6 is 2 times 3, we can put those together to have 54 is 2 times 3 cubed. This process is sometimes called making a factor tree, and pictured above are three possible ways for creating a factor tree for 54.

5.9 FACT FAMILIES II

Families: In Stage 3 we explored the fact families that connect addition and subtraction, and we saw how useful they are in understanding the interconnections between those two operations. Similar to what was done for addition and subtraction, group multiplication and division facts by families to gain a deeper understanding of them. For example, 3 x 4 = 12, 4 x 3 = 12, 12 / 3 = 4, and 12 / 4 = 3 form a fact family.

Multiplying and dividing are related: For the 3 x 4 = 12 fact family, visualize this interconnection using a rectangle that is 3 by 4. The area of this rectangle is 12, which is 3 times 4 or 4 times 3 – its width times its length. To get an area of 12 for a rectangle of width 3, the length must be 4. To get an area of 12 for a rectangle of length 4, its width must be 3. All these facts are tied together.

Multiplying and dividing undo each other: Let's continue to use the example of 3 times 4. If we start with 3 and multiply it by 4 we arrive at 12. If we then take 12 and divide it by 4, the result is back to 3. Multiplying by 4 and then dividing by 4 gets back to where we started.

Similarly, if we take 12 and divide by 4 the answer is 3. If we then multiply 3 by 4 the answer is 12, which is where we started. Dividing by 4 and then multiplying by 4 returns things to the beginning.

5.10 SINGLE-DIGIT DIVISION

You have prepared the way: Your work with skip counting, learning multiplication facts, and fact families will smooth the way for dividing both with and without remainders. All those skills should be well mastered and understood before starting in on general single-digit division.

No remainder: Division problems where the divisor goes in evenly without a remainder are mostly done by your child recognizing the corresponding multiplication fact. For example, if they are asked to divide 36 by 4, then remembering that 4 times 9 is 36 will get them straight to the answer. However, if that method doesn't work for a given problem, then they should use the next method.

Guessing and skip counting: Suppose your child is asked to divide 29 by 4. Your child will not find 29 among their multiplication facts involving 4, so they will want to find a result that is less than 29 that does work. Have your child guess at a multiple of 4 that is less than 29. They might guess 24, which is 4 times 6. They can then skip count forward until they bump into 29. In this case that will mean moving forward to 28, which is 4 times 7. Seeing that they have to stop there, they have their answer that 29 divided by 4 is 7 with a remainder of 1.

Checking the answer: Checking their answers is a good habit for your child to develop. In this last example, we think we found that 29 divided by 4 is 7 with a remainder of 1. Check this by multiplying 4 times 7 to get 28, and then adding 1 to get 29. So it all checks out!