Early Family Math

Stage 5 – I Can Count to 100!

Prerequisite: Can count to 100 comfortably and have a good sense of those quantities, especially using place value. Mental single-digit addition and subtraction is also solid.

Where You've Been

Your child can now count to 100! They can comfortably do mental single-digit addition and subtraction. They can also count or skip count up or down by any number, and tied to that skill is their ability to add or subtract a single-digit number with a double-digit number. They can compare two double-digit numbers, and they have a beginning sense of place value with 10's and 1's.

As their skip counting is improving, they are also developing skills with multiplying by 2, 3, 4, 5, and 10. The idea of even and odd numbers now makes a lot more sense to them.

Extend activities from earlier stages to these larger numbers: Stage 3: Shape Sums, Going Up Some More; Stage 4: War - Double Digit Add and Subtract, DiffTriangles and SumTriangles, Fix It, Island Hopping by 1's and 10's, Fill in the Blanks Comparison, Sum Square, and Addition Pyramid.

New Ideas in this Stage

- **Counting to 200** – Introduce the 100's place by looking at the numbers from 100 to 200.
- **Skip Counting to 100** – This is not new, but it is an important skill to reinforce.
- **Expanded Form and Place Value** – This is a foundational skill, so it will be reinforced further.
- **Double-digit Addition and Subtraction** – Skip counting will help make this seem effortless.
- **All Single-digit Multiplication** – It is time to fill in the missing gaps for 6, 7, 8, and 9.
- **Rectangle Area is Length x Width** – This is an important idea in its own right. This fact will also provide many opportunities for fun games and puzzles involving multiplication and factoring.
- **Factoring** – Your child will learn the beauty of how numbers break apart into factors. 1 is a unit. A number bigger than 1 only divisible by 1 and itself is prime. A number bigger than 1 not prime is composite. 3 squared is 3 x 3. 3 cubed is 3 x 3 x 3. And 3 raised to a power, means to multiply 3 by itself that many times - for example, 3 to the fourth is 3 x 3 x 3 x 3.
- **Factors, Divisors, and Multiples** – 3 divides evenly into 12. That makes 3 a factor or divisor of 12, and 12 a multiple of 3. 3 is a common factor of 12 and 15, and 12 is a common multiple of 4 and 6.
- **Single-digit Division** – Your child will learn division indirectly in the form of finding a missing factor in a multiplication problem.
- **Fact Families for Multiplication and Division** – We'll reinforce the connection between these two operations. For example, 2 x 5 = 10, 5 x 2 = 10, 10 / 2 = 5, and 10 / 5 = 2 form a fact family.

MENTAL MULTIPLICATION

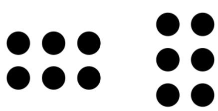
Prerequisite: Comfort adding/subtracting single-digit numbers, skip counting, and doubling

Introduction

These teaching methods provide structured strategies for learning single-digit multiplication. Your child should already be good at doubling any number, skip counting by any number, and multiplying by 5 and 10.

3 x 4 = 4 x 3

Your child is so familiar with addition that it's no surprise that 2 + 3 is the same as 3 + 2. Although not as obvious, the same is true for multiplication.



This illustration shows that two rows of three is the same as three rows of two – you're just changing your point of view! It doesn't matter which order you multiply two numbers, you get the same answer.

It's great that this cool observation means that your child needs to master only about half as many multiplication facts – once your child knows 3 x 4, they also know 4 x 3.

Squares

Just as addition twins are favorite addition math facts, squares are often favorites for multiplication. Learning these provides another foundation for learning other multiplication facts.

1 More or 1 Less

When combined with the other earlier skills, the strategy of using '1 more' or '1 less' is effective for calculating the remaining multiplication facts.

For example, 9 x 7 is one 7 less than 10 x 7. So 9 x 7 = 70 - 7 = 63. This works for all 9's.

Similarly, 3 x 7 is one more 7 than doubling 7, so 3 x 7 = 7 + 14 = 21. This works for all 3's.

Multiplying by 9

Although multiplying by 9 is covered by the last strategy, they are fun to learn in their own right. If you write the multiples of 9 in order, you'll see the tens digit is always one less than the number you are multiplying by and the ones digit plus the tens digit always adds up to 9!

PLACE VALUE AND ADDITION GAMES

***Prerequisite:* A sense of 2-digit place value and how it relates to addition and comparisons**

Making 100

Game

The setup: Each player has a sheet of paper with 7 rows and 3 columns. The columns are marked "10's," "1's," and "Running Total."

How to play: Each player's running total starts at 0. Roll a die or pick a random playing card from 1 to 9. Each player chooses to use this number in their 1's or 10's column for the current row. For example, if it is a 4, this can become 4 or 40. The chosen number is added to the running total.

How to win: A player that goes over the target of 100 "goes bust" and loses. If neither player goes bust, the one closer to 100 wins.

Variations

There are many options for this game:

- *Use a different target number*
- *Use fewer or more rows.*
- *Don't go bust if you go over the target. The closer player on either side wins.*
- *Use a fourth column of 100's to practice 3-digit numbers.*
- *Practice subtraction by starting at the target number and subtract down to 0.*

Stake Your Claim

Game

The setup: Have a paper with a number line from 0 to 99 to share.

How to play: On a turn, a player uses two random cards from 0 to 9, choosing the order of these two digits, to generate a number from 00 to 99, and then puts that number on their side of the number line.

How to win: The first player to get four numbers in a region without any of the opponent's numbers in between wins.

Variation

The game can also be played from 000 to 999.

PLACE VALUE, ADD, AND SUBTRACT

***Prerequisite:** A sense of 2-digit place value and how it relates to addition and comparisons*

Bonded Groups

Puzzle

13

7	9	9	6
6	4	4	7
2	5	11	2
6	1	7	5

There are two versions of these puzzles.

Version 1: The is the same as the Sum Groups puzzles in Stage 3, only now the target sums can be bigger. The boards can be any size, and here we use a 4 by 4 board. The target number is on the left, which is 13 in this case.

The challenge: Identify groups of connected numbers whose sum is the target (13)

Version 2: Here is an example of a 4 by 4 board with a target number of 20. As in Sum Groups, the board is filled with pairs and triples of numbers that add up to the target. However, now there will be one square not involved in any of those groups.

The challenge: The challenge is to find the single square that has that number. In this example, it is '5.'

20

7	9	7	4
8	4	4	16
12	5	9	6
13	7	7	7

Missing Numbers

Puzzle

How to create: Create these puzzles by taking a simple addition or subtraction equation and leaving out some of the digits. If you accidentally leave out too many numbers, that can open up a discussion of what all the possible solutions are – for example, if you start with 2 + 5 = 7 and leave out the first and third numbers, there are many solutions to ? + 5 = ?

Example: The following two problems are turned into Missing Number puzzles by leaving out a couple digits in each one.

```
  23    73         _3    7_
 +46   -46  --->  +46   -_6
  69    27         6_    27
```

Letter substitition puzzles: These missing number puzzles form a very understandable stepping stone to using simple variables. After getting comfortable with these puzzles, your child will be ready to do some Letter Substitution puzzles that are described later in this Stage.

MULTIPLICATION CARDS AND TABLES

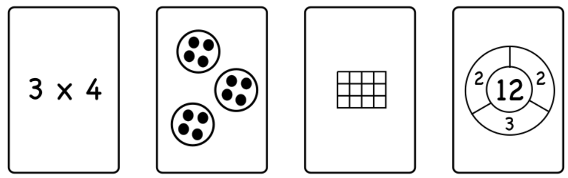
Prerequisite: Increasing comfort with single-digit multiplication for all numbers

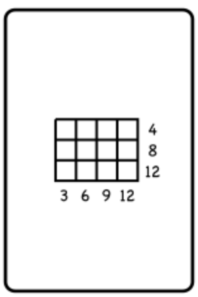
Make Multiplication Cards

Activity

Make a set of multiplication cards to practice these math facts while playing matching games your family played earlier: Stage 1 - Go Fish, Memory Challenge; Stage 2 - Bingo; Stage 3 - Hot Potato; and Stage 4 - Gin Rummy.

How to create: Hand draw four cards for each math fact. 1) the expression 2) groups of objects, 3) an array, and 4) the prime factorization. The four cards for 3 x 4 are:

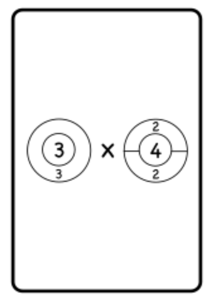




Options: One option is to include 3 x 4 and not include 4 x 3. This has the drawback that seeing 3 groups of 4 is different from seeing 4 groups of 3.

For array cards, put the skip counting numbers along one or both sides to help your child practice skip counting.

For expression cards, replace each number with the prime factorization symbol for the number. This makes it easier to see how the prime factorizations fit together when multiplying two numbers.



Revealing Products

Puzzle

X	5	3	7	8
2	10	6	14	16
9	45	27	63	72
8	40	24	56	64
5	25	15	35	40

How to create: Use a blank table with 4 product rows and columns. There are also groups of four missing numbers at the top and left sides – these will have some of the numbers, possibly with duplication, from 2 to 9. Fill in the table out of sight of your child, and then flip over or cover the numbers.

The challenge: Your child can ask to reveal, one at a time, up to 10 of the 16 product entries. The goal is to find the entries for the top and left sides before running out of turns.

Example: Imagine all cards are flipped over in this example. If your child chose to flip over the card that has the 63 under it, they would know it came from 7 and 9. Flipping over any other card in the same row or column as the 63 would show where the 7 and 9 are. Suppose the second card they flipped over is the 56. The third column must be 7, and also the second row is 9 and the third row is 8.

FACTORS AND MULTIPLES

Prerequisite: Increasing comfort with single-digit multiplication for all numbers

Cover Factor and Multiples

Game

The setup: Have a board of numbers from 1 to 24. There are two kinds of tokens – a single token reserved for "the last move," and a pile of other tokens.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24

How to play: The first player gets to pick any number and cover it with the last move token. After that, a player replaces the last move token with the other type of token and moves the last move token to any number that is a factor or multiple of the number from the last move.

How to win: The losing player is the one forced to cover the number 1.

Example: This board shows the middle of a game that started 10 -> 5 -> 15 -> 3.

Variations

As children get better at this game, they will discover rules governing reasonable first moves. The most basic rule is that the first move cannot be on a prime number in the upper half of the numbers.
Adjust the range of numbers to the players' skill level – 1 to 30, 1 to 48, or 1 to 60.

Nim With Factors

Game

The setup: Start with any number, say 20. Let your child decide whether to go first or second.
How to play: During their turn, a player may subtract any divisor of the current number from the number. For example, starting at 20, the first player can subtract 1, 2, 4, 5, or 10 for their first move.

How to win: The player forced to 0 loses.

Strategy

After your child becomes familiar with the game, encourage them to look for the remarkably simple strategy for allways winning - once they discover it, see if they can explain why it works.

FINDING PRIMES

Prerequisite: Increasing comfort with single-digit multiplication for all numbers

Sieve of Eratosthenes

Investigation

Children have fun putting in X's and watching primes fall through this sieve. This investigation creates opportunities for discovering many interesting properties of divisibility and primes.

Start with a number line numbered from 1 to 25 (or larger if space and patience allows).

Write the number 2 below itself. On this new line put X's below each multiple of 2.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	2		X		X		X		X		X		X		X		X		X		X		X	

Next, pull down the lowest number with no X's below it (3 in this case) and put it on the next line. Write the 3 and put X's on that line for all its multiples.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	2		X		X		X		X		X		X		X		X		X		X		X	
		3			X			X			X			X			X			X			X	

Keep pulling down numbers and marking their multiples.

When you are finished, you will have pulled down all the primes. Remember that 1 is a unit and not a prime!

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
	2		X		X		X		X		X		X		X		X		X		X		X	
		3			X			X			X			X			X			X			X	
				5					X					X					X					X
						7							X							X				
										11											X			
												13				17		19				23		

Questions

Discuss these questions with your child as they play with the sieve:

- *Why are primes the numbers the numbers that are pulled down?*
- *What is the last prime whose multiples you need to cross out? Why were the other primes not useful?*
- *For all the primes that were useful, which of their multiples produced new restrictions and which were not useful? Is there a pattern in that answer?*
- *If you had a number, say 53, which prime numbers would you need to divide it by to confirm that it's a prime?*

MIXED OPERATIONS

Prerequisite: Comfort with two-digit addition and subtraction, and one-digit multiplication

Mix It Up

Game

The setup: Use numbered cards from 1 to 25, or a range your child is comfortable with.

How to play: A card is selected at random and used as everyone's target number. That card is returned to the deck. Each player is dealt five cards to be used, in any order and with any operations, to get as close as possible to the target number.

Examples: Suppose the target number is 14, and you are dealt 3, 6, 12, 17, and 20. 17 - 3 or 20 - 6 work, but only use two cards. 20 - 12 + 6 uses three cards. 17 x (6 / 3) - 20 or 20 - (12 / (6 / 3)) use four cards, so that's an improvement, if you're trying to use all the cards. Can you find a way to use all five?

Scoring options

There are several options for scoring, and you may think of your own.

- *1 point to each player who hits the target. Total over several rounds.*
- *A player's score for a round is the difference between their result and the target. The scores are totaled over several rounds, and the lowest total score wins.*
- *A player earns twice as many points as the number of cards they use to reach the target; a player receives 5 points for hitting the target with help; and a player receives 6 points for helping someone hit the target.*

Parentheses Puzzles

Puzzle

The challenge: Take an expression, such as $2 + 7 \times 5 - 2 \times 2$, and add parentheses to it so that a target result, say 9, is obtained.

How to create: These are easy to create and tailor to you child's skill set. Take any equation, in our case $9 = (2+7) \times (5-2\times 2)$, and then remove the parentheses. It's as simple as that! Use the operations and numbers that your child is comfortable with. Use shorter expressions and fewer parentheses to make simpler puzzles.

MIXED OPERATIONS

Prerequisite: Comfort with two-digit addition and subtraction, and one-digit multiplication

Secret Operations

Activity

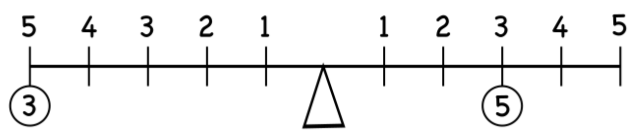
Near the end of Stage 4, the Sum Difference activity had one person think of two numbers and then challenge the other person to find the numbers by telling them the sum and difference of the numbers. Secret Ops uses the same idea, only now the challenger can use any two operations, such as multiplying and subtracting.

Examples: The challenger might say “Which two numbers have a product of 12 and a difference of 4?” You can extend this to three numbers, if you like - “Which three numbers have a product of 12 and a sum of 8?”

Lever Balance

Investigation

Levers: Use the lever principle to practice multiplication and addition. The principle says that the force exerted by a mass on one side of a lever is equal to the mass times its distance from the pivot point, the fulcrum. The forces on one side from several masses add up to give the total force. The total forces on the two sides must be equal for the lever to be in balance.



Examples: You have a 3-unit weight and a 5-unit weight to put on opposite sides of the fulcrum. Where should they be put to balance? The answer to this can be distances 5 and 3, but it can also be 10 and 6, or even larger answers like 15 and 9.

If you have a 3-unit and a 5-unit weight to put on one side of a lever, which weights can you put at which distances on the other side? What if the two weights are on different sides of the lever? This question continues the questions on the Make It Count page at the end of Stage 4.

MULTIPLYING AND MULTIPLES

Prerequisite: Comfort with single-digit multiplication

Beep

Game

The setup: Put the players in a circle. Start by identifying a group of numbers to use for a round of the game. Choose any group of numbers that would be fun or that provide practice with a concept. Some standard choice are:

- odd numbers or even numbers
- multiples of 3 (or some other number)
- multiples of 3 together with multiples of 7
- multiples of 3 that are not multiples of 5
- multiples of 3 together with numbers that have the digit 3 in them
- prime numbers

How to play: Going around the circle, the players take turns saying the numbers starting at 1. When a player has a number in the group, they must say 'beep' instead of the number. If a player fails to say beep, or says beep for an incorrect number, they're out.

How to win: The last player remaining in wins!

3 in a Row

Game

The setup: Use a deck of cards with Q's (as 0's), A's (as 1's), and 2-9, or use four sets of Number Cards from 0 to 9. Use a 4 by 5 grid on a paper with 20 spaces randomly filled out with multiples of 5 and 10. Have a set of tokens for each player.

How to play: Select a random card and put your token on that number times 5 or 10 - your choice. Once occupied, the other player cannot move there.

How to win: The first player to get 3 in a row wins.

Variations

The numbers 5 and 10 can be replaced by other pairs such as 2 and 4, or 3 and 6. These pairings help with practicing doubling strategies for multiplication. For example, if the player does not know 6 x 7, they can double 3 x 7.

MULTIPLYING AND TABLES

Prerequisite: Comfort with single-digit multiplication

War – Multiplication

Game

The setup: Remove the picture cards from a deck and split it evenly between two players. To give more focused practice, remove the A's and 10's as well.

How to play: Each player turns over two cards, multiplies them, and the player with the larger product wins those four cards. If the products are equal, two more cards are turned over and the winner gets to keep all eight cards.

How to win: The player with the most cards after playing for a set time is the winner.

Turning the Tables

Puzzle

Filling in a standard multiplication table is boring, and children quickly realize they can fill it using repeated addition rather than multiplication. To really practice multiplication, as well as practicing problem solving and factoring, create a mixed-up multiplication table.

How to create: Make these tables by moving the rows and columns around, and then leaving out most of the headings and entries in the middle.

X	5				6			
2								
		40						
				49				
	20					36		
		72						
			9					12
					48			

Example: Here is an example using headings of 2 through 9:

How to solve: Start with the distinctive entries.

The 20 forces its row to be multiplying by 4, and then the 36 makes its column be 9.

The 49 forces its column and row to be multiplying by 7. The 9 forces its column and row to be multiplying by 3.

Continue the detective work in this way and fill in the entries as the headings are discovered.

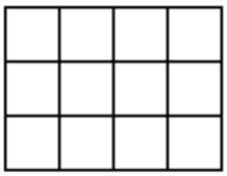
Level of difficulty: Increase or decrease the difficulty by leaving out more or less of the numbers. In this example, the '5' in the 5 column could have been left out – there must be a factor of 5 in 20, and that factor cannot come from 20's row because there is a 36 in that row.

Your child can make them: Challenge your child to make one of these puzzles for you. A lot of good thought can go into creating one of these!

RECTANGLE AREA

Prerequisite: Comfort with single-digit multiplication and double-digit addition

Introduction



The area of a rectangle is its length times its width. That dry statement can be made tangible to your child in at least two ways.

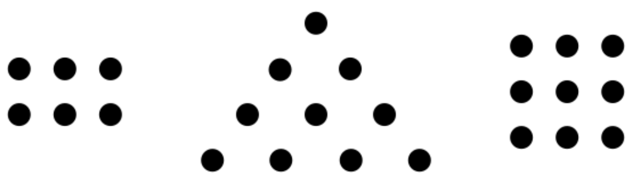
The first uses rectangles broken into an array of squares. The second uses number shapes to see how a quantity, such as 12, can be placed into an array – 3 by 4, 2 by 6, or 1 by 12. Playing with rectangle areas gives us an arena to play around with multiplication and factoring!

Number Shapes Revisited

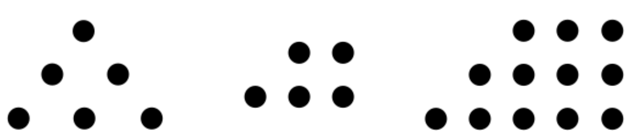
Investigation

Start with a large collection of small objects, such as raisins. For each number, investigate which rectangles and other shapes you can make with that many objects.

Rectangles: The sides of rectangles are values that evenly divide the number and multiply together to give the number. Making rectangles is a direct way to experience divisibility.



Unit, prime, composite: 1 is a unit, and can only be made with a 1 by 1 rectangle. The numbers, such as 5, that only have flat rectangles, are called primes. Numbers that are not a unit or a prime are called composite because they are composed of primes being multiplied together, such as 12 = 2 x 2 x 3. Numbers, such as 9, are called squares because one of their rectangles is a square – one rectangle for 9 is the 3 by 3 square.



Trapezoidal numbers: There are other shapes that are fun to investigate. For example, which numbers are trapezoidal? These are the numbers that can be represented as stair steps (where each level changes its length by 1)? If you include triangular numbers in this group, the answer is surprising – it is all numbers that are not a power of 2!

RECTANGLE AREA GAMES

Prerequisite: Comfort with single-digit multiplication and double-digit addition

The Paddock Game

Game

The setup: Each player gets a piece of graph paper.

How to play: For a player's turn, use two playing cards from 1 to 10 to determine the dimensions of a rectangle. If a player's paper has room, the rectangle may be placed anywhere its interior does not overlap with an existing rectangle. Once placed, its interior is lightly shaded and its area and dimensions are written on it. If there is no room, the turn is skipped.

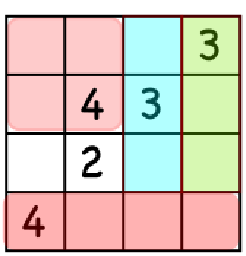
How to win: The player with the largest total wins.

Variations

For a normal piece of graph paper, this can be a long game – reduce the time by using half the paper or limiting the number of turns.

Divide Up the Box

Puzzle



The challenge: A rectangle, 4 by 4 or larger, with numbers in some of its squares, is to be divided into smaller rectangles. Each number must end up in a separate rectangle whose area is that number.

How to create: Out of the sight of your child, create these puzzles by first filling in the big rectangle with smaller rectangles. Next, place the area in each rectangle. Lastly, give your child the big rectangle with only the numbers.

Solving strategies

To solve these puzzles, look first at areas that are prime numbers – their shapes are tightly constrained.
Next, consider regions that are boxed in. In this puzzle the upper "4" must relate to the upper left 2 by 2 square. Also, the upper right corner must be used in a vertical 3 by 1 rectangle.

FEEL THE POWER

Prerequisite: Comfort multiplying single-digit numbers

A Definition and a Rule

The definition: Just as 4×2 is a quick way to write $2+2+2+2$, so 2^4 is a quick way to write $2 \times 2 \times 2 \times 2$. It's much easier to say and understand the phrase "two to the fourth," than to say "two times two times … "

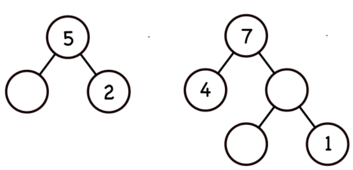
Special names: The second power, 4^2 for example, can be said *four squared*, and the third power, 4^3 for example, can be said *four cubed*.

The rule: When powers of the same number are multiplied, a simple rule governs how to simplify the result – add the powers. For example, if you do $4^2 \times 4^3 = (4\times4)\times(4\times4\times4) = 4^5$, we have two fours multiplied by three fours, so the result is five fours being multiplied.

Caution: Note that this rule for adding exponents only works when it is the same number being taken to a power – you can't easily simplify $3^2 \times 5^3$.

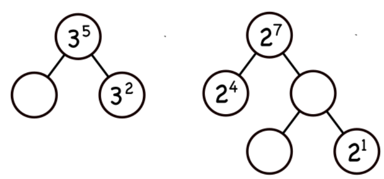
Reuse Old Addition Activities

Reuse puzzles and games: Because powers add up when powers are being multiplied, any of our old games and puzzles involving addition can be used to practice multiplying numbers that are powers. Some examples of these old addtition activities are: Stage 3 - Shape Sums and Sum Groups; Stage 4 - Enclosed Sums, SumTriangles, and Fix It.



On the left are two examples in Stage 3 – Shape Sums.

On the right are the same examples for Shape Products, using multiplication instead of addition.



Working with powers will becomesroutine and just as easy as the original addition problems.

Variations: If your child enjoys these problems and wants some extra challenge, start involving more than one number being raised to a power. For example, if you multiply $(4^2 \times 3^3) \times (4^5 \times 3^2)$ you can apply the rule separately to the powers of 4 and the powers of 3 and get the result $4^7 \times 3^5$.

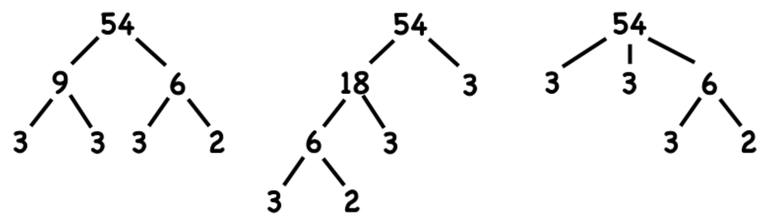
DISCOVERING PRIME FACTORIZATIONS

Prerequisite: Comfort doing single-digit multiplication

Factor Trees

Investigation

The model for factor trees is an extension of Shape Products from the previous Feel the Power page. The goal in creating a factor tree is to reduce a number to its prime factors. Many things can be learned about a number in the process of constructing a factor tree.



Example: Start with a number, say 54. This can be broken down several ways. One way is 9 x 6, another is 18 x 3, and yet another is 3 x 3 x 6. Each produces a start to a factor tree.

Each of these trees ultimately produces the same primes on its leaves. In each case we end up with 2 x 3 x 3 x 3, but look at the different ways of getting there!

Questions

After doing some examples like these, your child may naturally start asking some questions.

- *Why do some trees have more levels than others?*
- *Why are some trees broader than others?*
- *Why do the leaves always stop at primes?*
- *Why do the leaves always have the same list of primes, perhaps with rearrangement?*

Fundamental Theorem: This last question is a really big topic, and is called the Fundamental Theorem of Arithmetic. It says a number has exactly one way of being written as a product of primes!

Why is that so important? It shows that primes are the multiplicative building blocks of numbers, and once you have found one way to build a number, that is the only way. If you know that 54 = 2 x 3 x 3 x 3, then there is no way, using whole numbers, to write 54 = 5 x ?? . The uniqueness of prime factorizations is at the heart of a lot of beautiful number theory.

FACTORING WITH PRIMES

Prerequisite: Comfort doing single-digit multiplication

Practice Prime Factorizations

Activity

Do prime factorizations in order when you travel or have time on your hands. This also provides practice with talking about powers. Knowing prime factorizations with ease will be helpful in many things to come, such as working with fractions. Have fun with this and don't push your child beyond their comfort level.

Recital

The recital goes like this:

1. *is a unit*
2. *is a prime*
3. *is a prime*
4. *is 2 squared*
5. *is a prime*
6. *is 2 x 3*
7. *is a prime*
8. *is 2 cubed*
9. *is 3 squared*
10. *is 2 x 5*
11. *is a prime*
12. *is 2 squared x 3*
13. *is a prime*
14. *is 2 x 7*
15. *is 3 x 5*
16. *is 2 to the fourth*
17. *is a prime*
18. *is 2 x 3 squared*
19. *is a prime*
20. *is 2 squared x 5*
21. *is 3 x 7*
22. *is 2 x 11*
23. *is a prime*
24. *is 2 cubed x 3*

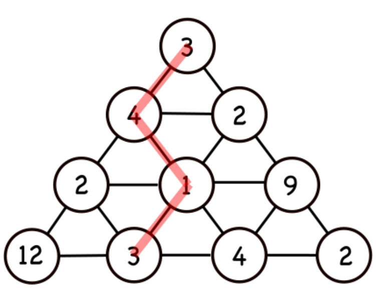
If your child stumbles, help them figure it out rather than simply giving them the answer.

Product Pyramid

Puzzle

These puzzles are the multiplicative version of the Addition Pyramids seen in Stage 4. You are supplied with a target number and a pyramid of numbers.

The challenge: The challenge is to find a path of connected numbers down the pyramid so the product of the selected numbers is the target.



36

In this pyramid the target is 36, and the red lines indicate the path that works.

These puzzles are easier if you start by doing the prime factorization of the target. Because 36 = 2 x 2 x 3 x 3, these prime factors must be picked up along the path, and this helps guide the search.

Knowing about prime factorizations also makes it much easier to create these puzzles.

ADDING AND SUBTRACTING

Prerequisite: A sense of 2-digit place value and how that relates to addition and subtraction

100 Laughs

Game

The setup: Use a deck of cards with Q's (as 0's), A's (as 1's), and 2-9's. Set the target number at 100. Four random cards are chosen in order and used to make a pair of 2-digit numbers, a shared resource.

How to play: Each player is dealt 14 random cards face up. Players alternate turns. During a turn, a player must use exactly two of their cards to place on top of two of the four cards. The player gets one point if the current two two-digit numbers add up to the target. The name of the game comes from the optional action of a player laughing each time they succeed in getting the target amount.

How to win: When the cards are all used up, the player with the most points wins.

Variations

- *Having a target of 100 is good for practicing number bonds for 10. However, other targets are useful for variety and practicing other number bonds.*
- *Give players fewer or more than 14 cards.*
- *Use subtraction together with a smaller target number.*

5-Card Draw to a Target

Game

The setup: Choose a target number, say 100.

How to play: Each player picks up five random cards from 0 to 9. Two 2-digit numbers are made out of these numbers, the fifth card is unused. The two numbers are added and the player closest to the target wins a point for that round.

How to win: The highest number of points after a fixed number of rounds wins.

Variations

One option is to use three-digit numbers, a target number of 1000, and each player receives seven cards. Another option is to use subtraction with a smaller target number.

LETTER SUBSTITUTIONS

Prerequisite: A sense of 2-digit place value and how that relates to addition and subtraction

Letter Substitution

Puzzle

The setup: In these puzzles, single digits are replaced by letters. At first glance, these puzzles seem to be the same as the 'Missing Number' puzzles from earlier in this Stage. However, the use of letters provides more interesting opportunities for problem solving. If your child is comfortable with the Missing Number puzzles, you should transition to these puzzles.

The use of letters in these puzzles follows three rules:

Three rules

- *A given letter is always the same digit from 0 to 9*
- *The leftmost digit of a number is never 0*
- *Different letters must be different digits*

How to create: Take an ordinary addition or subtraction problem and replace one or more of the digits. Use the same letter when replacing the same digit. In this example, 6 is replaced by 'A' in both places.

```
  23         23    B3
 +46  --->  +4A   +4A
 ---        ---   ---
  69         A9    A9
```

```
  B     B     A     A
 +8    +B    +A    +2
 ---   ---   ---   ---
  C     8    C4    BC

  A     A     B    BA
 +B   +BB   +AB   +BB
 ---   ---   ---   ---
 AC    A7    BA   CAB
```

Special puzzles: The circumstances of this type of puzzle allow for the creation of interesting problem-solving challenges. These take a bit of designing, but the result is some fun puzzles.

Notice that the values of the letters do not carry over from puzzle to puzzle. The 'B,' which has value 1 in this first puzzle, has value 4 in the second.

SHAPES INSIDE SHAPES

Prerequisite: Curiosity and persistence to find patterns involving shapes

Filling Regions with Shapes

Investigation

Suppose you have an 8 by 8 chessboard and a collection of 1 by 2 tiles. Finding a way to exactly cover the chessboard with 32 of these 1 by 2 tiles is simple enough.

Remove corners: Let's start playing with removing squares from the chessboard. If you remove one corner of the chessboard, you know immediately that you can no longer cover the chessboard with tiles because the tiles will cover an even number of squares, and there are now 63 squares. Okay, remove two corners to make an even number of remaining squares – can you cover it now? The answer depends on which two corners you remove. Why? What if you no longer restrict yourself to removing corners, what happens?

Learn from smaller examples: One important lesson in dealing with questions like these is to learn from smaller problems. Try these questions on a 4 by 4 or 6 by 6 board first.

Variations

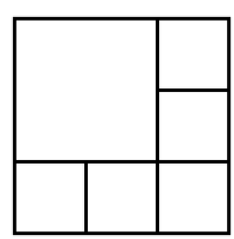
Branch out to using other shapes to fill the board. Play around with filling it with 1 by 3 tiles or with 3 squares in an L shape. What patterns and rules do you discover with these? What other shapes might be interesting to look at?

Filling Squares with Squares

Investigation

The challenge: In which ways can you fill a square with other squares, where the squares need not all be the same size? The question is: What are all the numbers of squares that are possible? For those that are possible, is there an easy way to describe how to do it?

Let your child play with it over many days without any hurry to get an answer. Here is a diagram showing how 6 is possible.



Variations

What happens if you only allow squares of certain sizes, such as 1 by 1, 2 by 2, and 3 by 3? What happens when filling other figures with figures that have the same shape? For example, use figures that are regular triangles (triangles with all their sides the same length). Which figures are interesting to investigate in this way?

MULTIPLYING AND MULTIPLES

Prerequisite: Comfort multiplying single-digit numbers and skip counting to 100

The Product Game

Game

The setup: Use a shared piece of paper filled out as shown.

How to play: The first player moves a token onto any number from 1 to 9 in the 1-9 squares. The second player puts another token on one of the 1-9 squares and claims the product in the 6 by 6 grid. From then on, each player chooses to move either of the two tokens and claims the product (if they can).

How to win: The first player with 3 squares in a row wins.

Variations: Mix up the product numbers to give your child better practice identifying the products. Also, see the Stage 5 Bonus Material for designs of larger boards with larger ranges.

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

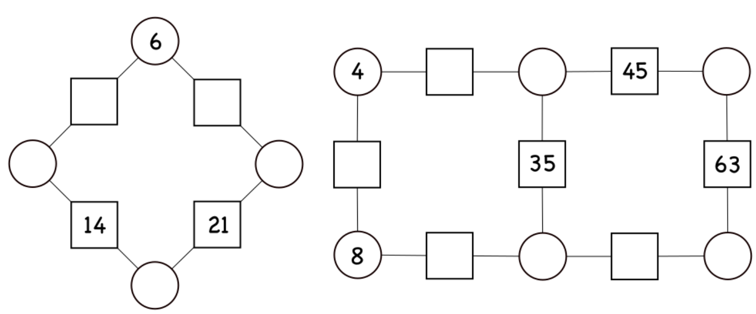
1	2	3	4	5	6	7	8	9

Island Hopping with Products

Puzzle

These puzzles have islands (circles and squares) connected by bridges (lines). If there are two circles on either side of a square, then the square holds the product of the two circles.

The challenge: Fill in the missing numbers.



How to create: Make these puzzles by filling in the circles, then filling in the squares, and finally removing some of the numbers before giving it to your child.

In addition to practicing multiplication, these puzzles can be structured to practice common factors as well. In the first puzzle, the only number, other than 1, that divides 14 and 21 is 7, so that is the number in the bottom circle.

ADD, SUBTRACT, AND MULTIPLY

Prerequisite: Comfort with two-digit addition and subtraction, and one-digit multiplication

Counting Neighbors

Game

The setup: Use three dice and an 8 by 8 board of numbers from 1 to 64.

How to play: A player rolls the dice and uses addition, subtraction, multiplication, and division to make any unmarked number on the board. The player marks this square and receives one point for the square plus one more point for each marked square that it touches, including diagonally. If a player cannot make a play, any other player who finds a play can claim that score.

How to win: Play five or more rounds, with the largest score winning.

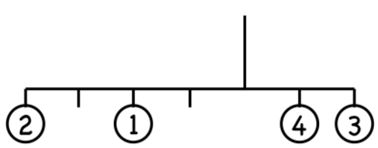
Variations

Some game options are to use a fourth die, and to use a smaller or larger board.

Making a Mobile

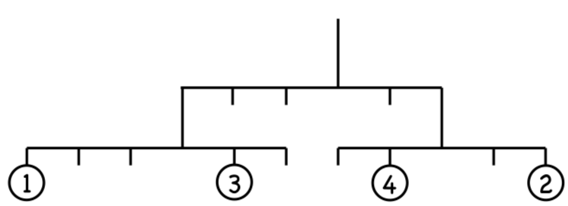
Puzzle

The setup: You are given some weights and a design for a mobile that has some attach points. The challenge is to put at most one weight per attach point so the mobile will balance along every arm. Assume the wires are weightless. Each arm in the mobile is a lever that needs balancing, so these puzzles are an extension of the Lever Balance puzzle given earlier in this Stage - practice those puzzles before starting these.



Simple example: Start with the simplest mobiles, which are just levers in the air. Here is a solution for putting the weights from 1 to 4 on this mobile to balance it. This works because 2 x 4 + 1 x 2 = 4 x 1 + 3 x 2.

More complicated example: Use the total of the weights below it to balance each side of the top wire (1 + 3) x 3 = (4 + 2) x 2.



Go to the Stage 5 Bonus Material for more examples and a longer discussion of mobiles.

ADD, SUBTRACT, AND MULTIPLY

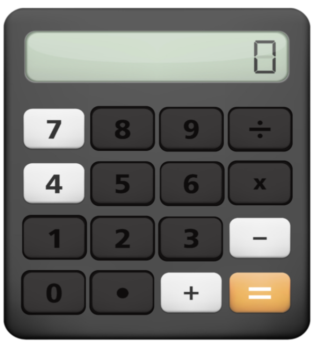
Prerequisite: Comfort with two-digit addition and subtraction, and one-digit multiplication

Limited Calculators

Puzzle

The setup: Suppose you have a calculator that is badly broken and you are challenged to produce some result on the calculator. This is easy to play orally whenever you have a spare moment. Here are some examples to get you started.

Example: Suppose you had a calculator with +, -, x, and /, but only one working number key, the 4. Could you get the result 21? If so, what is the fewest number of steps you would need? Suppose you could use 4 at most four times - which numbers could you produce? Suppose you had to use the 4 exactly four times. Play around with having other single keys and creating other results.



Example: Suppose your calculator could only add 4 or 7. Which numbers could you produce?

Example: Suppose the calculator only had 4 or 7, but now it can add and subtract. Which additional numbers could you produce?

These are the same activities we've seen previoiusly in other settings, such as with the pan balance.

Example: Suppose you only had a 1 key and could only add or double. For example, 2 x (2 x 1) + 1 is 5. What other numbers can you create?

Example: A fun challenge is the four 4's challenge. Suppose you had a calculator with only a 4 key, with all the operations working. Starting at 1, how many numbers can you create if you must use exactly four 4's in your work on the calculator?

FACTORS AND MULTIPLES

Prerequisite: Can multiply single-digit numbers and is getting better at factoring numbers

Grabbing Factors

Game

The setup: Use a board with a 4 by 6 grid of numbers from 1 to 24.

How to play: On a turn, a player chooses a number that is uncovered and has at least one factor uncovered – the player gets the selected number and the other player gets any or all of the uncovered factors (their choice as to how many). Play alternates until there are no legal numbers left.

How to win: The players add up their numbers and the higher sum wins.

Tax Collector solitaire: This can also be played as a solitaire puzzle, called Tax Collector. The one player selects each number and the tax collector gets all the available factors. Play continues until the player no longer has a legal move. At that point, the tax collector receives the remaining numbers. The goal is to have as large a sum as possible – bigger than the tax collector when that's possible.

Variations

Set the range of numbers for the skill level of the players. It could be 1 to 12, or as high as 1 to 60.

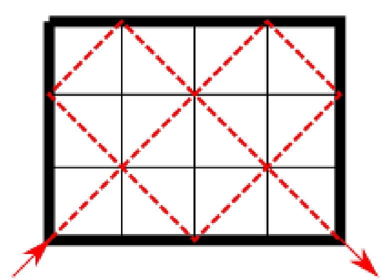
Bouncing Billiard Ball

Investigation

Imagine a billiard table that has a pocket in each of the four corners. When a ball bounces off the side of the table, it bounces away at the same angle it came in at.

This investigation looks at the question: If we shoot a ball at a 45 degree angle from one corner, where will it end up?

The answer depends on the size of the table. This is what happens on a 3 by 4 table.



After playing with several different sizes of tables, challenge your child to predict what the answer is in advance for new sizes. Starting in the bottom left corner, which corner will be hit first and how many bounces will it take?

FACTORS COUNT

Prerequisite: Can multiply single-digit numbers and is getting better at factoring numbers

Double or Nothing

Game

The setup: Players start the game by secretly picking 5 distinct numbers larger than 20 and less than 121. After all selections are made, they are written where all can see them.

How to play: Using Number Cards or some other device, a random number from 1 to 20 is created. That number is repeatedly doubled until either someone's number is hit for the first time or the number becomes bigger than 120.

How to win: The first player to have all five numbers hit is the winner.

Strategies for selecting numbers

It is a bad idea to pick a number, such as 46, that is not a power of 2 times some number between 1 and 20 – it will never get hit. Some numbers with lots of factors of 2, such as 32, are more likely to be hit because more starting numbers can get to them.

Variations

You can triple the number each time instead of doubling it. You can double it and add 1 each time. For younger players, select numbers above 10 and not above 60, and select a random number from 1 to 10.

War With Factors

Game

The setup: Have two sets of cards, say from 1 to 25.

How to play: Play the standard game of war with these cards, only now the winner is the card that has more factors. For example, 12 beats 16 because 12 has 6 factors (1, 2, 3, 4, 6, and 12) while 16 has 5 factors (1, 2, 4, 8, and 16). The holder of the winning card must be able to correctly list the factors to win the cards – otherwise, the cards get shuffled back into each player's draw pile. As with standard War, when there is a tie, the next cards are turned over and the winner receives all the cards.

Variations

You can play that the smaller number of factors wins. You can count the total of just the prime factors rather than all the factors. You can play that prime powers (numbers that are a power of a prime) beat other numbers.

MULTIPLICATION BOARD GAMES

Prerequisite: single-digit multiplication and skip counting

Crossing the Volcano

Game

The setup: Use a 100-chart with the 36 squares on the four edges colored gray. Use playing cards with picture cards removed or use Number Cards from 1 to 10.

How to play: On a turn, if you pick a 1 you can claim any odd number; if you pick any other number, you can claim any multiple of it. If you claim a number, your opponent cannot claim it. The aim is to make a path from one edge to the opposite edge, in either direction. You do not need to claim the squares in the order of your path.

Variations

You can either play that diagonal connections are okay or not okay. Another option is to include picture cards – if you get one of these, you can put in a blocked square that cannot be included in either person's path.

Checkers Math

Game

The setup: This game is lightly inspired by checkers. Each player has 10 counters. The counters are numbered from 1 to 10, with the "10" counter marked with 10 and 11. The counters start on the end rows of a 100-chart - one player on squares 1 to 10 and the other on squares 91 to 100.

How to play: Initially, counters can only move "forward" one row onto any multiple of the number(s) on the marker they choose – for the player starting on 1 to 10, forward means larger numbers, and for the player starting on 91 to 100, forward means smaller numbers. Once a counter has made it all the way across the board, it becomes a king and can then move forward or backward one row after that. An opponent's piece is taken by landing on it. A player's piece cannot double up with another of the same player's pieces.

How to win: You win by taking all your opponent's pieces.

Variations

For younger players, shorten the board to use the first 6 rows – the numbers from 1 to 60. A child who does not know all the multiples yet can use skip counting to figure out the moves.

INTERESTING PRODUCTS

Prerequisite: single-digit multiplication and skip counting

Multiplication Bingo

Game

The setup: Each player starts with a 4 by 4 grid of numbers that are possible multiplication products – these numbers can either be randomly assigned or carefully chosen by the player.

How to play: To start, two cards are dealt and put face up on the table. If either player has the product of those two numbers, they cover it. From then on, the players take turns taking the top card from the draw pile and choosing which of the two cards to replace. All players who have a match with the product cover it.

How to win: The first player to get 4 in a row wins.

Cross Products

Puzzle

This multiplication puzzle is either 3 by 3 involving each of the numbers 1 to 6 exactly once, or 4 by 4 involving the numbers 1 to 8 exactly once.

The challenge: Fill in some of the squares, two numbers for each row and each column, so that the product of the numbers in a row is the number marked to the far left and the product of the numbers in a column is the number marked above the column. Some rows or columns may not be marked – if so, there is no constraint on the product of those rows or columns.

Solving examples: Solve this puzzle by finding columns and rows where you can identify the two numbers. The 30 column must have 5 and 6, and the 10 row must have 2 and 5. Next, the 12 column must have 3 and 4 and the 4 row must have 1 and 4. The rest follows quickly.

		30	12
4			
10			

		30	12
4	1		4
10	2	5	
		6	3

	7		40	18
5				
21				
32				

	7		40	18
		2		6
5	1		5	
21	7			3
32		4	8	

The 5 row in this puzzle must have 1 and 5, the 21 row has 3 and 7, the 32 row has 4 and 8, and the top row has, by elimination, 2 and 6. The 7 column has 1 and 7, the 40 column has 5 and 8, the 18 column has 3 and 6, and the second column has 2 and 4. Put this together for the solution.

How to create: As is often the case with these puzzles, the adult can make them by filling in the numbers on the inside of the puzzle first, writing down the products, and then removing all the interior numbers.